Fractional Differential Systems: A Fuzzy Solution Based on Operational Matrix of Shifted Chebyshev Polynomials and Its Applications

Ali Ahmadian, *Member, IEEE*, Soheil Salahshour, and Chee Seng Chan, *Senior Member, IEEE*

***Abstract*—In this paper, a new formula of fuzzy Caputo fractional-order derivatives $(0 < v \leq 1)$ in terms of shifted Chebyshev polynomials is derived. The proposed approach introduces a shifted Chebyshev operational matrix in combination with a shifted Chebyshev tau technique for the numerical solution of linear fuzzy fractional-order differential equations. The main advantage of the proposed approach is that it simplifies the problem alike in solving a system of fuzzy algebraic linear equations. An approximated error bound between the exact solution and the proposed fuzzy solution with respect to the number of fuzzy rules and solution errors is derived. Furthermore, we also discuss the convergence of the proposed method from the fuzzy perspective. Experimentally, we show the strength of the proposed method in solving a variety of fractional differential equation models under uncertainty encountered in engineering and physical phenomena (i.e., viscoelasticity, oscillations, and resistor–capacitor (*RC*) circuits). Comparisons are also made with solutions obtained by the Laguerre polynomials and the fractional Euler method.**

***Index Terms*—Chebyshev polynomials, fuzzy fractional differential equations (FFDEs).**

I. Introduction

FRACTIONAL differential equation (FDE) is one of the most important branches of fractional calculus, as it has proven to be very suitable for modeling memory effects of various engineering applications, compared with the traditional integer-order models [1]–[5]. Particularly, in the field of dynamical systems and control theory, many works had utilized an FDE to study the anomalous behavior of dynamic systems. For instance, Caputo and Mainardi [6] formulated a mathematical model of viscoelasticity connecting the Hooke elastic element and the Maxwell viscoelastic element based on a fractional-order model; Bagley and Torvik [7] used fractional calculus to construct stress–strain relationships for viscoelastic materials; Diethelm and Luchko [8] employed an FDE to describe the process of the solution of gas in a fluid; and Lin and Lee [9] and Tavazoei [10] studied adaptive fuzzy sliding-mode control to synchronize two different uncertain fractional-order time-delay chaotic systems.

Fig. 1. Fort Point Channel Tunnel in Boston, 12 tons of concrete fell on a car due to creep in the epoxy that was used to support anchoring bolts for the ceiling panels.

In the fuzzy domain, the study of fuzzy differential equations (e.g., in this paper, we consider fuzzy fractional differential equations—FFDEs) had created a suitable setting for mathematical modeling of real-world problems, in which uncertainties or vagueness penetrate. This is in order to avoid the repetition of *Big Dig ceiling collapse* incident, occurred on July 10, 2006.[1] Based on the investigation, concrete ceiling panels in the Boston's Fort Point Channel were hung using bolts embedded in epoxy (a type of polymer). Over time, the bolts pulled out of the epoxy causing a three-ton panel to crash on the roadway. The panel landed on a car carrying a young couple, killing the female passenger and injuring the male driver. The epoxy used was a viscoelastic material that deforms over time when a force is applied to it, until it reaches an equilibrium state (creep). In this case, the concrete panel weighed too much for the epoxy and caused it to deform to the point of failure. The failure of this panel set off a chain reaction that eventually led to 12 tons of concrete falling to the roadway, as illustrated in Fig. 1. Had engineers consider uncertainty and, at the same time, fully understood viscoelasticity, this incident might have been avoided.

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A. Ahmadian and C. S. Chan are with the Centre of Image and Signal Processing, Faculty of Computer Science and Information Technology, University of Malaya, 50603 Kuala Lumpur, Malaysia (e-mail: ali.ahmadian@um.edu.my; cs.chan@um.edu.my).

S. Salahshour is with the Young Researchers and Elite Club, Islamic Azad University, Mobarakeh Branch, Mobarakeh 8481914411, Iran (e-mail: Soheil.Salahshour@gmail.com).

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[1] https://en.wikipedia.org/wiki/Big_Dig_ceiling_collapse

In one of the earliest works, Agarwal, Lakshmikantham, and Nieto [11] took the initiative and introduced fuzzy fractional calculus (i.e., FFDEs) to handle the fractional-order systems with uncertain initial values or uncertain relationships between parameters. They considered the solution of FFDEs under Riemann–Liouville H-differentiability. Using this new concept, Arshad and Lupulescu [12] were pioneers to study the existence of solution for a class of fuzzy fractional integral equations, as well as the properties of Riemann–Liouville integral of fuzzy-valued functions under Riemann–Liouville generalized H-differentiability, which is a direct generalization of the fractional Riemann–Liouville derivative using the Hukuhara difference. Afterward, Allahviranloo, Salahshour, and Abbasbandy [13] employed the Riemann–Liouville generalized H-differentiability in order to solve the FFDEs and presented some new results under this notion.

Mazandarani and Vahidian Kamyad [14] introduced a Caputo-type fuzzy fractional derivative to solve FFDEs. Recently, Salahshour, Allahviranloo, and Abbasbandy [15] used fuzzy Laplace transform in order to solve such problems analytically, which was then followed up by Mazandarani and Najariyan [16], [17] who introduced fuzzy Laplace transform under type-2 fuzzy fractional differentiability. However, there are some flaws associated with these aforementioned solutions. In the former, as highlighted in [18], there is a limitation in the Hukuhara difference, as it leads solutions with increasing length of their support. In the latter, the Riemann–Liouville derivative requires a quantity of the fractional derivative of unknown solution at the initial point, and in the last one, the methods under type-2 fuzzy sets theory will lead to an increase in the computational cost, although it is closer to the originality of the model. It has very recently been introduced by Lupulescue [19], a generalization of the Hukuhara difference to develop a theory of the fractional calculus for interval-valued functions, which was a continuation of the concept proposed in [20].

At the same time, orthogonal functions have received noticeable consideration in dealing with various problems. The main advantage in using this method is that it simplifies the problem alike in solving a system of algebraic equations, leading to simplify the original problem. In addition, it is proven that accurate approximation can be achieved with relatively few degrees of freedom. The most popular orthogonal functions are block-pulse, Legendre, Laguerre, Jacobi, and Chebyshev. Saadatmandi and Dehghan [21] introduced a shifted Legendre operational matrix for fractional derivatives and applied it with *tau* and *collocation* methods to find numerical solutions of multiterm linear and nonlinear FDEs subject to initial conditions. The author of [22]–[25] derived a new formula expressing explicitly any fractional-order derivatives of shifted Chebyshev polynomials of any degree in terms of shifted Chebyshev polynomials themselves and used it with spectral methods to solve multiterm linear and nonlinear FDEs. An extension to the *tau* method to handle multiorder FDE variable coefficients using the shifted Legendre Gauss-Lobatto quadrature is studied in [26]. Esmaeili, Shamsi, and Luchko [27] introduced a *collocation* technique to obtain the spectral solution with Müntz polynomials. Motivated by these results, the authors of [28]–[30] presented the spectral *tau* method for numerical solutions of FDEs using various types of orthogonal polynomials. Comparing different types of orthogonal polynomials, we observe that all of them have a common characteristic in that the function approximation is a series approximation. The Chebyshev approximation, however, has an additional distinct characteristic over the Walsh, block-pulse, and Laguerre polynomials, as such the function approximation is simultaneously an (almost) uniform approximation, i.e., the errors are distributed nearly uniformly in the time interval of interest. This characteristic is of importance in cases where it is desirable that the error involves in the approximation is not concentrated in certain portions in the time interval of interest, but it is uniformly distributed over the interval [31]. However, most of these solutions are based on a rigorous framework, that is, they are often tailored to deal with specific applications and are generally intended for small-scale fuzzy fractional systems.

In this paper, our aim is to derive an explicit formula for a fuzzy fractional-order derivative in terms of shifted Chebyshev polynomials introduced by Doha, Bhrawy, and Ezz-Eldien [25], but in the fuzzy Caputo sense, i.e., we introduce a suitable way to approximate a fuzzy solution for linear FFDEs, using the shifted Chebyshev polynomials functions based on the fuzzy-like residual of the problem. To this end, the Chebyshev operational matrix (COM) is introduced in the derivation of the proposed method. It is worth mentioning here that, although other orthogonal bases such as the Jacobi polynomials [32] present nice stability properties and are very useful in approximation, Chebyshev is more advantageous for practical computations on account of its intrinsic numerical stability [31]. In addition, to get the acceptable accuracy using the orthogonal polynomials such as the Laguerre and Jacobi, it needs to find the optimized values of Jacobi or Laguerre parameters α and β, while the Chebyshev or Legendre polynomials are not entangled with this issue. As a resultant, it decreases the computational time. Moreover, only a small number of shifted Chebyshev polynomials are required to acquire convincing results, which has been proven in Section III with applications in viscoelasticity, oscillations, and resistor–capacitor (*RC*) circuits.

In summary, the major contributions of this paper are the following.

1) We introduce a shifted COM of a fuzzy fractional Caputo's derivative, which is based on the Chebyshev tau method to solve a linear FFDE numerically. The main characteristic of this new technique is that it converts an FFDE to a simple fuzzy algebraic equation. Then, one of the advantages of the tau method lies in its accuracy for a given number of unknowns. For problems whose solutions are sufficiently smooth, they exhibit exponential rates of convergence/spectral accuracy, and this has motivated us to utilize this method to solve the FFDE model.
2) In the fuzzy algebraic system, the first row of the coefficients matrix is zero, and the last row is replaced by a suitable formulation of the initial conditions. Such a solution has several advantages, for example, being a) nondifferentiable, b) nonintegral, and) can be easily implemented on a computer system, because its structure is dependent on the matrix operations only.
3) To date, and to the best of our knowledge, the proposed approach (i.e., using COM) has not been explored widely

to solve fuzzy mathematical models in terms of fractional cases. Simulation results indicate that the proposed solution is not only feasible, but also provides valuable information for a number of applications in engineering and physical phenomena (i.e., viscoelasticity, oscillations, and *RC* circuits).

The rest of this paper is structured as follows. Section II presents an introduction on the tau method for solving linear FFDEs using the COM of a fuzzy fractional Caputo's derivative. The upper bound of approximation error between the exact and the proposed fuzzy solution is derived to confirm that the approximate solution is bounded and the method is convergent. In Section III, we report our numerical findings and demonstrate the strength of the proposed scheme by considering numerical simulations. In addition, in this section, we present some mathematical models in the fuzzy sense to evaluate our technique when dealing with fuzzy real phenomena. Finally, conclusion and some recommendations for future work are drawn in Section IV.

II. Proposed Method

In this section, we detailed the proposed spectral solution for the FFDE using the COM. Particularly, a spectral tau method [22], [25] is presented to find the fuzzy approximate solutions of the linear FFDE in terms of the operational matrices of the fuzzy fractional Caputo's derivative, based on the shifted Chebyshev polynomials in the interval [0,1]. This is unlike Ahmadian, Suleiman, Salahshour, and Baleanu [32] who employed the *tau* method for the solution of the FFDE. In their work, the corresponding Jacobi functions are not a good choice because in our paper, we deal with two crisp ordinary differential equation systems transfer from the assumed FFDE under fuzzy differentiability, which should be solved together to achieve the fuzzy approximate solution. If we employ the Jacobi functions, we will have to find the optimized values of the Jacobi parameters to obtain the lowest errors, and this will lead to high computational time. With this, we deduce that the Jacobi functions are not a cost-effective choice for the fuzzy cases.

In contrast, the main advantage of our new technique using the shifted Chebyshev polynomials in the interval [0,1] is that only a small number of the shifted Chebyshev polynomials are required and the good accuracy will be acquired in one-time program running. Thus, it greatly simplifies the problem and reduces the computational costs. The solution is expressed as a truncated Chebyshev series, and therefore, it can be easily evaluated for arbitrary values of time using any computer program without any computational effort. In this paper, we concentrate our study on nonperiodic fuzzy problems, and therefore, Chebyshev series expansions fit best in this practical requirement [33].

A. Chebyshev Approximation of a Fuzzy Function

First, we introduce some notations that will be used later in the paper.

1) $L_p^{\mathbb{E}}(a,b), 1 \le p < \infty$ is the set of all fuzzy-valued measurable functions f on $[a,b]$, where $\|f\|_p = (\int_0^1 (d(f(t)),0))^p dt)^{\frac{1}{p}}$ for $p < \infty$.
2) $C^{\mathbb{E}}(a,b)$ is a space of fuzzy-valued functions, which are continuous on $[a,b]$.
3) $C_n^{\mathbb{E}}(a,b)$ indicates the set of all fuzzy-valued functions, which are continuous up to order n.

In order to obtain the approximation of a fuzzy function according to the shifted Chebyshev polynomials (37), we present some key definitions. For more details of the approximation of a fuzzy-valued function, see [34] and [35].

Let us consider $u \in C(J,\mathbb{E}) \cap L^1(J,\mathbb{E})$, and that the shifted Chebyshev polynomial $T_i^*(x)$ is a real-valued function over [0,1]; then, we aim to find the fuzzy approximate function, $u_N(x) : \mathbb{R} \mapsto \mathbb{E}$, which can be stated similar to the definition presented in the crisp context [22], [23], as

$$u(x) \simeq u_N(x) = \sum_{i=0}^{+\infty}{}^{\bullet} c_i \odot T_i^*(x)$$

where the fuzzy coefficients c_i are given as

$$c_i = \frac{1}{h_i}\int_0^1 u(x) \odot T_i^*(x) \odot w(x) dx, \quad i = 0, 1, \ldots \quad (1)$$

in which $w(x) = \frac{1}{\sqrt{x-x^2}}$, $T_i^*(x)$ has the similar definition to the shifted Chebyshev polynomials presented in the Appendix, and $\sum\bullet$ denotes a sum with respect to $\oplus$ in $\mathbb{E}$. In the remainder of this section, we will describe the procedure that leads to achieve the fuzzy approximate solution and validate the method by analyzing the convergence and several numerical cases.

Practically, only the first $(N+1)$ terms of the shifted Chebyshev polynomials are considered, and therefore, we have

$$u(x) \simeq u_{N+1}(x) = \sum_{i=0}^{N}{}^{\bullet} c_i \odot T_i^*(x) = C^T \odot \Phi(x). \quad (2)$$

The fuzzy shifted Chebyshev coefficient vector C^T and shifted Chebyshev function vector $\Phi(x)$ are described as

$$\begin{aligned} C^T &= [c_0, c_2, \ldots, c_N] \\ \Phi(x) &= [T_0^*(x), T_1^*(x), \ldots, T_N^*(x)]^T. \end{aligned} \quad (3)$$

We can specify the fuzzy approximate function $u_{N+1}(x)$ based on the lower and upper functions as follows.

Definition 2.1: Let $u \in C(J,\mathbb{E}) \cap L^1(J,\mathbb{E})$; the approximation fuzzy-valued function $u_{N+1}(x)$ in the parametric form is

$$u(x,r) \simeq u_{N+1}(x,r) \quad (4)$$

$$= \left[\sum_{i=0}^{N} c_{i,-}(r) T_i^*(x), \sum_{i=0}^{N} c_{i,+}(r) T_i^*(x),\right], \quad 0 \le r \le 1. \quad (5)$$

B. Operational Matrix of Fuzzy Caputo Derivative

In this section, generalization of the operational matrix of Chebyshev functions is derived based on the Caputo derivative. Afterward, the error function of the fuzzy Caputo fractional derivative operator is provided to depict that the fuzzy approximate function is in good agreement with the fuzzy Caputo differentiable function. For more details on the case of fuzzy and nonfuzzy context, see [22], [25], and [36].

Theorem 2.1 (see [24] and [25]): The approximation of the Caputo derivative by means of shifted Chebyshev polynomials for $0 < v \le 1$ can be represented as

$$ {}^{c}D^{v}T_i^*(x) = i\sum_{k=\lceil v\rceil}^{i}(-1)^{i-k}\frac{(i+k-1)!2^{2k}k!}{(i-k)!(2k)!\Gamma(k-v+1)}x^{k-v} $$

$$ i = \lceil v\rceil, \lceil v\rceil + 1, ..., N. $$

As stated in [24], the COM based on their Chebyshev polynomials is given by

$$ D^{v}\Phi(x) \simeq D^{(v)}\Phi(x) \tag{6} $$

where $D^{(v)}$ is the $(N+1)$-square operational matrix of the Chebyshev function based on Caputo differentiability and $D^{v}\Phi(x) \in C(J)$. Therefore, using (6) and Theorem 2.1, we can approximate the Caputo's derivative of the fuzzy approximation function as

$$ \begin{aligned} D^{v}u(x) &\simeq D^{(v)}u_{N+1}(x) \\ &= \sum_{i=0}^{N}{}^{\bullet}c_i \odot D^{(v)}T_i^*(x) = C^T \odot D^{(v)}\Phi(x). \end{aligned} \tag{7} $$

C. Error Analysis

Now, in order to evaluate the error bound of the fuzzy approximate function, we introduce the following lemma. This lemma shows that the approximation converges of the Chebyshev functions to function f in a deterministic case.

Lemma 2.1: Let the function $f : [x_0, 1] \to \mathbb{R}$ is N times continuously differentiable for $x_0 > 0, f \in C^N[x_0, 1]$, and $\mathbf{T}^N = \text{Span}\{T_i^*(x)\}_{i=0}^N$. If $f_N = C^T\Phi(x)$ described in (41) is the best approximation to f from $\mathbf{T}^N$, then the error bound can be presented as

$$ \| f(x) - f_{N+1}(x)\|_w \le \frac{MS^N}{(N)!}\sqrt{\pi} $$

where $M = \max_{x\in[x_0,1]} f^{(N)}(x)$, $S = \max\{1-x_0, x_0\}$, and $\| f \|_w = (\int_0^1 f(x)^2 w(x)dx)^{1/2}$ [37].

Proof: See Appendix A. ∎

Theorem 2.2 provides an upper error bound for the fuzzy approximation function based on the shifted Chebyshev polynomials. Based on this theorem, it can be confirmed that the fuzzy approximate function is convergent to the main function.

Theorem 2.2: Let the function $u \in C(J,\mathbb{E}) \cap L^1(J,\mathbb{E})$ be continuously fuzzy differentiable for $x_0 > 0, u \in C([x_0, 1], \mathbb{E})$, and $\mathbf{T}^N = \text{Span}\{T_0^*(x), T_1^*(x), \ldots, T_N^*(x)\}$. If $u_N = C^T \odot \Phi(x)$ is the best fuzzy approximation to $u(x)$ from $\mathbf{T}^N$, then the error bound can be represented as

$$ D(u(x), u_N(x)) \le \frac{Q(r)S^N}{(N)!}\sqrt{\pi} $$

where $Q(r) = \max_{x\in[x_0,1]}\{M_-(r), M_+(r)\}$, $S = \max\{1 - x_0, x_0\}$, and $r \in [0, 1]$.

Proof: Considering Lemma (2.1), the proof is straightforward. ∎

Now, let us first define E_v as

$$ E_v = |D^v\Phi(x) - D^{(v)}\Phi(x)| = [E_{0,v}, E_{1,v}, \ldots, E_{N,v}]^T $$

where

$$ E_{k,v} = |D^vT_k^*(x) - \sum_{j=0}^{N} D_{\text{kj}}^{(v)}T_j^*(x)|, \quad k = 0, 1, \ldots, N. $$

Lemma 2.2: Let the error function of the Caputo fractional derivative operator for Chebyshev polynomials $E_{i,\alpha} : [x_0, 1] \to \mathbb{R}$ is $N+1$ times continuously differentiable for $0 < x_0 \le x$, $x \in (0, 1]$. In addition, $E_{i,v} \in C^{N+1}[x_0, 1]$ and $v \le N+1$, and therefore, the error bound can be represented as

$$ \| E_{i,v} \|_w \le \frac{|\Gamma(i+1)|}{|\Gamma(1-v)|}\frac{MS^{N+1}}{(N+1)!}x_0^{-v}\sqrt{\pi}. $$

Proof: See Appendix B. ∎

Thus, the maximum norm of error vector E_v is achieved as

$$ \| E_v \|_\infty \le \frac{|\Gamma(N+2)|}{|\Gamma(1-v)|}\frac{MS^{N+1}}{(N+1)!}x_0^{-v}\sqrt{\pi} $$

where $\| E_v \|_\infty = \max_{i=0,1,\ldots,N}|E_{i,v}|$.

D. Case Study: Walkthrough of Our Proposed fuzzy Fractional Differential Equation Solution

The main goal of this section is to show the steps of our proposed method in solving the linear FFDE. We derive a fuzzy-like residual of the approximate problem based on Chebyshev functions. Then, using the tau method, $N+1$ fuzzy algebraic linear equations are derived and are solved by finding the unknown fuzzy coefficients of the approximate fuzzy solution. Here, the problem has reduced from an original fuzzy fractional problem to a fuzzy algebraic linear equations system, which is much easier to handle.

Consider the following linear FFDE:

$$ \begin{cases} ({}^{c}D_{0^+}^{v}y)(x) + y(x) = f(x), & 0 < v \le 1 \\ y(0) = y_0 \in \mathbb{E} \end{cases} \tag{8} $$

where $y \in C(J,\mathbb{E}) \cap L^1(J,\mathbb{E})$ is a continuous fuzzy-valued function, ${}^{c}D_{0^+}^{v}$ indicates the fuzzy Caputo's fractional derivative of order v, and $f(x) : [0, 1] \mapsto \mathbb{E}$.

Let $\langle .,. \rangle_{\mathbb{E}}$ denote the fuzzy-like inner product over the weighted $X_{\mathbb{E}} = L_w^2(J,\mathbb{E})$. It can be presented in the r-cut form as follows:

$$ [\langle p, q\rangle_{\mathbb{E}}]^r = [\langle p_-^r, q_-^r\rangle_w, \langle p_+^r, q_+^r\rangle_w] $$

where $\langle p_-^r, q_-^r\rangle_w$ and $\langle p_+^r, q_+^r\rangle_w$ are inner products over the weighted $X_{\mathbb{R}} = L_w^2(J,\mathbb{R})$. For more details, see Appendix D (Lemma 4.1).

As in a typical tau method [37], [38], we generate N fuzzy linear equations by applying

$$ \langle R_N(x, r), T_i^*(x)\rangle_E = \tilde{0}, \quad i = 0, 1, \ldots, N-1, \quad r \in [0, 1] \tag{9} $$

where $\langle R_N(x, r), T_i^*(x)\rangle_E = [(FR)\int_0^1 R_N(x, r) \odot T_i^*(x) \odot w(x)dx]$, and R_N is the *fuzzy-like residual operator* for (8), which is defined in the matrix operator form of

$$ R_N(x, r) = [\underline{R}_N(x, r), \overline{R}_N(x, r)] $$

where

$$\begin{cases} \underline{R}_N(x,r) = \underline{C}^T(r)(D^{(v)}\Phi(x) + \Phi(x)) - \underline{F}^T(r)\Phi(x) \\ \overline{R}_N(x,r) = \overline{C}^T(r)(D^{(v)}\Phi(x) + \Phi(x)) - \overline{F}^T(r)\Phi(x). \end{cases} \tag{10}$$

Now, regarding to the relation (9) and using the following statements:

$$\begin{aligned} y_N(x,r) &= \sum_{j=0}^{N} c_j(r) T_j^*(x) \\ f_k(r) &= \langle f(x,r), T_k^*(x)\rangle_w, \quad k = 0,1,\ldots,N-1 \\ f(r) &= (f_0(r), f_1(r), \ldots, f_N(r), y_0(r))^T \\ y(0,r) &= y_0(r) \end{aligned}$$

we have

$$\begin{aligned} &\langle R_N(x,r), T_k^*(x)\rangle_E \\ &= \Big[\int_{P_1} \underline{R}_N(x,r)T_k^*(x)w(x)dx + \int_{P_2} \overline{R}_N(x,r)T_k^*(x)w(x)dx, \\ &\int_{P_1} \overline{R}_N(x,r)T_k^*(x)w(x)dx + \int_{P_2} \underline{R}_N(x,r)T_k^*(x)w(x)dx\Big] \\ &= \Big[\int_{P_1} (\underline{C}^T(r)D^{(v)}\Phi(x) + \underline{C}^T(r)\Phi(x) - \underline{F}^T(r)\Phi(x))T_k^*(x)w(x)dx \\ &+ \int_{P_2} (\overline{C}^T(r)D^{(v)}\Phi(x) + \overline{C}^T(r)\Phi(x) - \overline{F}^T(r)\Phi(x))T_k^*(x)w(x)dx, \\ &\int_{P_1} (\overline{C}^T(r)D^{(v)}\Phi(x) + \overline{C}^T(r)\Phi(x) - \overline{F}^T(r)\Phi(x))T_k^*(x)w(x)dx \\ &+ \int_{P_2} (\underline{C}^T(r)D^{(v)}\Phi(x) + \underline{C}^T(r)\Phi(x) - \underline{F}^T(r)\Phi(x))T_k^*(x)w(x)dx\Big] \\ &= \tilde{0}, \quad k = 0,1,\ldots,N-1. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\langle R_N(x,r), T_k^*(x)\rangle_E = \\ &\Big[\int_{P_1} \underline{C}^T(r)D^{(v)}\Phi(x)T_k^*(x)w(x)dx \\ &+ \int_{P2} \overline{C}^T(r)D^{(v)}\Phi(x)T_k^*(x)w(x)dx, \\ &\int_{P_1} \overline{C}^T(r)D^{(v)}\Phi(x)T_k^*(x)w(x)dx \\ &+ \int_{P_2} \underline{C}^T(r)D^{(v)}\Phi(x)T_k^*(x)w(x)dx\Big] \\ &+ \Big[\int_{P_1} \underline{C}^T(r)\Phi(x)T_k^*(x)w(x)dx + \int_{P_2} \overline{C}^T(r)\Phi(x)T_k^*(x)w(x)dx, \\ &\int_{P_1} \overline{C}^T(r)\Phi(x)T_k^*(x)w(x)dx + \int_{P_2} \underline{C}^T(r)\Phi(x)T_k^*(x)w(x)dx\Big] \\ &= \Big[\int_{P_1} \underline{F}^T(r)\Phi(x)T_k^*(x)w(x)dx + \int_{P_2} \overline{F}^T(r)\Phi(x)T_k^*(x)w(x)dx, \\ &\int_{P_1} \overline{F}^T(r)\Phi(x)T_k^*(x)w(x)dx + \int_{P_2} \underline{F}^T(r)\Phi(x)T_k^*(x)w(x)dx\Big] \end{aligned}$$

for $k = 0,1,\ldots,N-1$ and $r \in [0,1]$. Therefore, we can rewrite the above results in a compact form as

$$\begin{aligned} \int_0^1 C^T(r)D^{(v)}\Phi(x)T_k^*(x)w(x)dx + \int_0^1 C^T(r)\Phi(x)T_k^*(x)w(x)dx \\ = \int_0^1 F^T(r)\Phi(x)T_k^*(x)w(x)dx \end{aligned}$$

for $k = 0,1,\ldots,N-1$ and $r \in [0,1]$.

Regarding (2) and (7), we have

$$\begin{aligned} &\int_0^1 D^{(v)}y_N(x,r)T_k^*(x)w(x)dx + \int_0^1 y_N(x,r)T_k^*(x)w(x)dx \\ &= \int_0^1 f(x,r)T_k^*(x)w(x)dx, \quad k = 0,1,\ldots,N-1. \end{aligned}$$

Using the definition of fuzzy-like inner product, we have

$$\begin{aligned} &\left\langle D^{(v)}y_N(x,r), T_k^*(x)\right\rangle_E + \langle y_N(x,r), T_k^*(x)\rangle_E \\ &= \langle f(x,r), T_k^*(x)\rangle_E, \quad k = 0,1,\ldots,N-1 \end{aligned} \tag{11}$$

and $r \in [0,1]$. Then, in order to acquire the approximation $y_N(x,r)$ using the shifted Chebyshev tau approximation, we should find the unknown vector $C^T = [\underline{C}^T(r), \overline{C}^T(r)]$. Therefore, (11) can be stated as follows

$$\begin{cases} \sum_{j=0}^{N} c_j(r)\left[\left\langle D^{(v)}T_j^*(x), T_k^*(x)\right\rangle_w + \left\langle T_j^*(x), T_k^*(x)\right\rangle_w\right] \\ = \langle f(x,r), T_k^*(x)\rangle_w,\ k = 0,1,\ldots,N-1, \\ \qquad j = 0,1,\ldots,N,\ r \in [0,1] \\ \sum_{j=0}^{N} c_j(r)T_j^*(0) = y_0(r). \end{cases} \tag{12}$$

Then, using the matrix form and their defined elements, described in Appendix C, (12) can be written in the following matrix form:

$$(\mathfrak{A} + \mu\mathfrak{B})\boldsymbol{C} = \boldsymbol{f}. \tag{13}$$

Finally, system (13) can be solved based on the following lower–upper representation by any direct or numerical method [39], [40]:

$$\begin{cases} \underline{(\mathfrak{A} + \mathfrak{B})\boldsymbol{C}} = \underline{\boldsymbol{f}} \\ \overline{(\mathfrak{A} + \mathfrak{B})\boldsymbol{C}} = \overline{\boldsymbol{f}}. \end{cases}$$

III. Applications of Fuzzy Fractional Differential Model

In this section, we bring forth the technical correctness of the proposed method with some numerical simulations in terms of few possible uses in engineering applications. The absolute errors of the problems in different conditions are provided to demonstrate the effectiveness of the COM with the tau method based on the Caputo-type fuzzy fractional differentiability of order $0 < v \leq 1$. In addition, in order to verify the reliability and accuracy of the proposed method, our numerical results are compared with the fuzzy fractional Euler method [14] and the tau method with the Laguerre operational matrix (LOM) [28] as the representative of the orthogonal functions (i.e., Walsh functions, Block-Puls functions, and Laguerre functions), which have common characteristics in that the derived operational matrices algorithms are similar.

Note that the accuracy of the methods is compared by computing the absolute errors $\underline{E}(t,r) = |\underline{y}_a(t,r) - \underline{Y}_e(t,r)|$, $\overline{E}(t,r) = |\overline{y}_a(t,r) - \overline{Y}_e(t,r)|$ (for a constant t and various values of r), where $Y_e(t,r) = (\underline{Y}_e(t,r), \overline{Y}_e(t,r))$ is the known exact solution and $y_a(t,r) = (\underline{y}_a(t,r), \overline{y}_a(t,r))$ is the approximate solution. In this regard, the absolute errors of the proposed method, the tau method with Laguerre functions, and the fractional Euler method are E_c, E_l, and E_u, respectively.

A. Application 1—Viscoelasticity

This application shows a possible use of the fuzzy fractional differential model in the field of viscoelasticity. As a definition, viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. This results in time-dependent behavior, which means that a material's response to deformation or force may change over time. Typical engineering materials have the same response to a force or deformation no matter how fast you apply the force/deformation or how long the force/deformation is present. It is very important for engineers to understand viscoelasticity if they are going to design devices that use or interact with polymers or biological materials so that the Big Dig ceiling collapse incident as explained in Section I could be avoided.

Formally, we consider viscoelasticity under uncertainty represented by fuzzy-valued functions. Let us consider the relationships between stress and strain for solids (Hooke's law) and for Newtonian fluids (Newton's law), respectively, as follows:

$$\begin{cases} \sigma(t) = Ee(t) \\ \sigma(t) = \eta \dfrac{d}{dt} e(t). \end{cases} \tag{14}$$

In (14), E and η stand for the spring' constant and the viscosity, respectively. On the other hand, noting that stress is proportional to the zeroth derivative of strain for solids and to the first derivative of strain for fluids, it is natural to suppose that for "intermediate" materials, stress may be proportional to the stress derivative of "intermediate" (noninteger) order:

$$\sigma(t) = \eta D_t^v e(t), \quad (0 < v \le 1) \tag{15}$$

where E and v are material-dependent constants.

The Hooke's law (14) is a one-parameter model, while the Scott Blair law (15) is a two-parameter model (i.e., the parameters arc η and v), which can be further generalized by adding further terms on both sides, containing arbitrary-order derivatives of stress and strain. This leads to a three-parameter generalized Voigt model

$$\sigma(t) = Ee(t) + \eta D^v e(t) \tag{16}$$

which describes the motion of a rigid plate immersed in a Newtonian fluid.

Now, in order to study the mentioned problem in a real case, we apply the fuzzy initial value e_0, the fuzzy-valued function $\sigma(t)$, and the concept of Caputo's H-differentiability for fractional derivative of $({}^cD_{0+}^v e)(t)$ and generalized H-differentiability [18] for first-order derivative of $e(t)$, $e'(t)$.

Let us consider the fuzzy model of the motion of a rigid plate immersed in a Newtonian fluid as follows:

$$\begin{cases} {}^cD_{0+}^v e(t) + e(t) = t^4 - \dfrac{1}{2}t^3 - \dfrac{3}{\Gamma(4-v)}t^{3-v} + \dfrac{24}{\Gamma(5-v)}t^{4-v} \\ e(0,r) = [-1+r, 1-r],\ 0 < v \le 1,\ 0 \le r \le 1, \quad t \in [0,1] \end{cases} \tag{17}$$

in which $e \in C(J,\mathbb{E}) \cap L^1(J,\mathbb{E})$ is a continuous fuzzy function, ${}^cD_{0^+}^v$ indicates the fuzzy Caputo's fractional derivative of order v, and $E = \eta = 1$.

According to the definition of ${}^c[1-v]-$differentiability and Theorem 4.1, we have

$$\begin{cases} ({}^cD_{0+}^v e_-)(t,r) + e_-(t,r) = t^4 - \dfrac{1}{2}t^3 - \dfrac{3}{\Gamma(4-v)}t^{3-v} \\ \qquad\qquad + \dfrac{24}{\Gamma(5-v)}t^{4-v} \\ e_-(0,r) = -1+r,\ 0 < v \le 1,\ 0 \le r \le 1, \quad t \in [0,1] \end{cases} \tag{18}$$

and

$$\begin{cases} ({}^cD_{0+}^v e_+)(t,r) + e_+(t,r) = t^4 - \dfrac{1}{2}t^3 - \dfrac{3}{\Gamma(4-v)}t^{3-v} \\ \qquad\qquad + \dfrac{24}{\Gamma(5-v)}t^{4-v} \\ e_+(0,r) = 1-r,\ 0 < v \le 1,\ 0 \le r \le 1. \quad t \in [0,1] \end{cases} \tag{19}$$

with the exact solution as

$$\begin{cases} e_-(t,r) = (-1+r)E_{v,1}[-t^v] + \int_0^t (t-x)^{v-1} \\ \qquad E_{v,v}[-(t-x)^v]\sigma(x)dx,\ 0 \le r \le 1 \\ e_+(t,r) = (1-r)E_{v,1}[-t^v] + \int_0^t (t-x)^{v-1} \\ \qquad E_{v,v}[-(t-x)^v]\sigma(x)dx,\ 0 \le r \le 1 \end{cases} \tag{20}$$

where $\sigma(x) = x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-v)}x^{3-v} + \frac{24}{\Gamma(5-v)}x^{4-v}$.

Let $N = 2$ and taking into account (18) and (19), we may write the approximate solution and the right-hand side as

$$e(t,r) \simeq e_{N+1}(t,r) = \sum_{i=0}^{2} c_i(r)T_i^*(t) = C^T(r)\Phi(t)$$

$$g(t) \simeq \sum_{i=0}^{2} g_i T_i^*(t) = G^T\Phi(t).$$

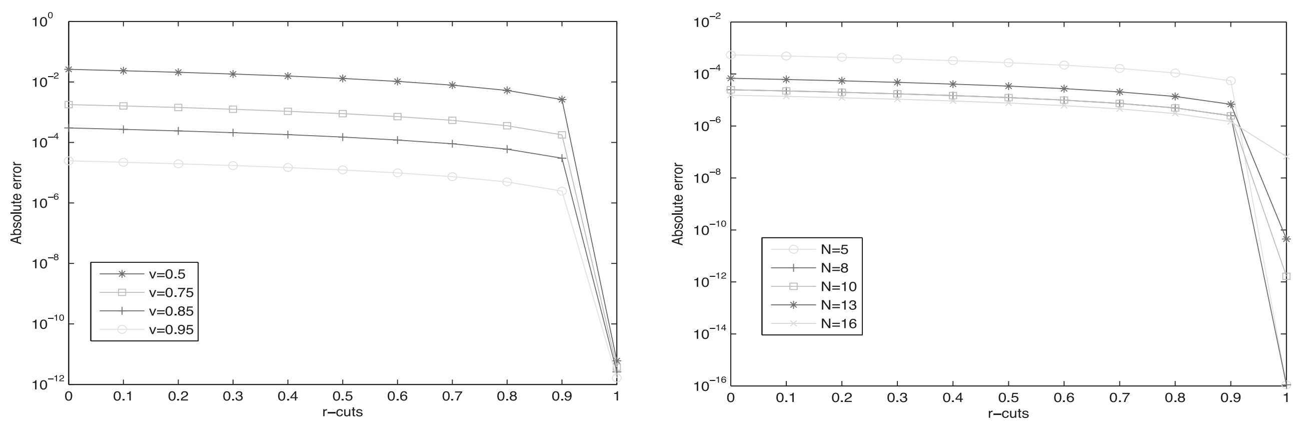
Here, we have

$$D^{0.85} = \begin{pmatrix} 0 & 0 & 0 \\ 1.7949 & 0.4682 & -0.1851 \\ -0.1221 & 5.6769 & 1.1000 \end{pmatrix}, \quad G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}.$$

Using (9), (12), and (13), we can derive a fuzzy algebraic linear equation system, and solving the aforesaid system, the following values for the unknown fuzzy coefficients c_i^r for the lower and upper bound of the fuzzy approximate function

TABLE I
COMPARISON OF THE ABSOLUTE ERRORS (E_c, E_l, AND E_u) FOR VISCOELASTICITY WITH $\alpha = 0.85$

r	$\underline{Y}_e(1;r)$	$\underline{E}_c(1;r)$	$\underline{E}_l(1;r)$[28]	$\underline{E}_u(1;r)$[14]	$\overline{Y}_e(1;r)$	$\overline{E}_c(1;r)$	$\overline{E}_l(1;r)$[28]	$\overline{E}_u(1;r)$[14]
0	0.11876	5.11991e-4	1.03903e-3	3.49858e-1	0.88123	5.13990e-4	1.03903e-3	2.89274e-1
0.1	0.15689	4.60692e-4	9.35134e-4	3.46828e-1	0.84310	4.62691e-4	9.35134e-4	2.92303e-1
0.2	0.19501	4.09393e-4	8.31230e-4	3.43799e-1	0.80498	4.11392e-4	8.31230e-4	2.95332e-1
0.3	0.23313	3.58094e-4	7.27326e-4	3.40770e-1	0.76686	3.60093e-4	7.27326e-4	2.98361e-1
0.4	0.27126	3.06794e-4	6.23422e-4	3.37741e-1	0.72873	3.08794e-4	6.23422e-4	3.01391e-1
0.5	0.30938	2.55495e-4	5.19518e-4	3.34712e-1	0.69061	2.57495e-4	5.19518e-4	3.04420e-1
0.6	0.34750	2.04196e-4	4.15615e-4	3.31683e-1	0.65249	2.06196e-4	4.15615e-4	3.07449e-1
0.7	0.38563	1.52897e-4	3.11711e-4	3.28653e-1	0.61436	1.54897e-4	3.11711e-4	3.10478e-1
0.8	0.42375	1.01598e-4	2.07807e-4	3.25624e-1	0.57624	1.03597e-4	2.07807e-4	3.13507e-1
0.9	0.46187	5.02993e-5	1.03903e-4	3.22595e-1	0.53812	5.22988e-5	1.03903e-4	3.16537e-1
1	0.50000	9.99757e-9	9.54645e-7	3.19566e-1	0.50000	9.99757e-9	9.54645e-7	3.19566e-1



(a) $N = 10$ (b) $v = 0.95$

Fig. 2. Viscoelasticity: Absolute errors of the proposed method, $\underline{E}_c(1;r)$ (a) for different values of Caputo derivative v and (b) for different values N.

are obtained:

$$c_{0,-}(0.1) = -0.4458,\ c_{1,-}(0.1) = 0.5135,\ c_{2,-}(0.1) = 0.1992$$
$$c_{0,+}(0.1) = 0.6734,\ c_{1,+}(0.1) = -0.0274,\ c_{2,+}(0.1) = 0.0593.$$

It can be seen that the coefficient $c_1^{0.1}$ does not satisfy the fuzzy condition for specific cut. Therefore, in this situation, we consider fuzzy weak solution. In fact, by changing the lower and upper values, a new approximate solution can also be obtained. Noticing that, however, this strategy will lead to more errors, which is one of the main differences between our proposed point of view and conventional approaches that investigated the solution by just using the lower and upper approximations without considering the fuzzy condition. Therefore, we can rewrite

$$e_-(t,0.1) = \begin{pmatrix} -0.4458 & -0.0274 & 0.0593 \end{pmatrix} \begin{pmatrix} 1 \\ -1+2t \\ 1-8t+8t^2 \end{pmatrix}$$

$$e_+(t,0.1) = \begin{pmatrix} 0.6734 & 0.5135 & 0.1992 \end{pmatrix} \begin{pmatrix} 1 \\ -1+2t \\ 1-8t+8t^2 \end{pmatrix}.$$

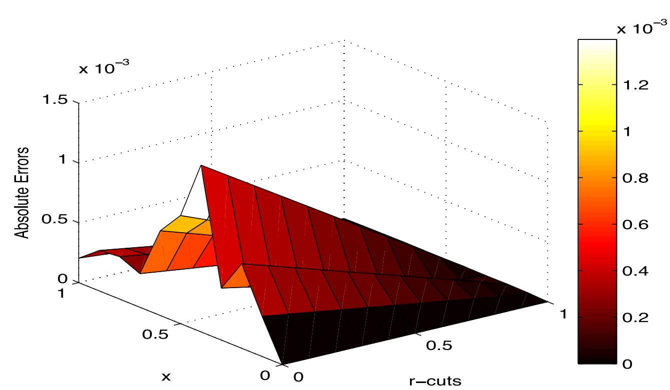
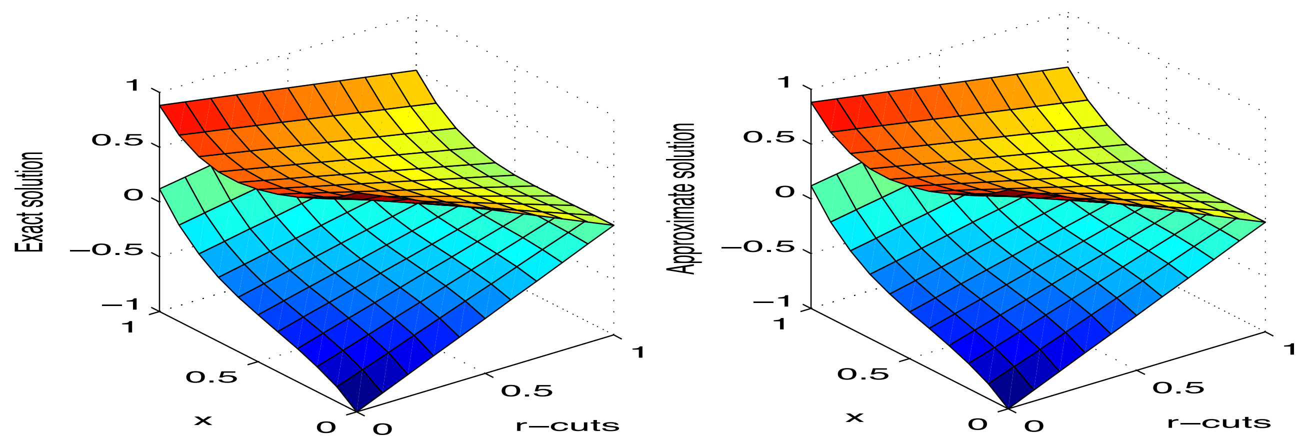


Fig. 3. Viscoelasticity: $\underline{E}_c(t;r)$ for $r \in [0,1]$ and $t \in [0,1]$ with $v = 0.85$ and $N = 10$.

Indeed, we just change the position of lower and upper functions for coefficients. In fact, $T_i^*(t)$ do not have the same sign in their domains. Therefore, we just check at the end of approximation that whether $e_-(t,0.1)$ is less than $e_+(t,0.1)$ or not. In fact, we do it for obtaining better solution, i.e., lower errors.

Remark 3.1: In general, the analytical solution of an FFDE is hard or impossible to obtain, even for the simplest FFDE ${}^cD_{0+}^v y(t) = -y(t)$. Thus, it is important and challenging to find



(a) (b)

Fig. 4. Profiles of (a) the exact solution and (b) the fuzzy solution of the motion model of a rigid plate immersed in a Newtonian fluid with $v = 0.85$ and $N = 8$.

some more suitable and better accuracy methods in the study of numerical simulations of FFDE due to various applications of FDE discovered in physical and engineering fields in recent years. However, this issue is even a fresh and prominent branch in FDE field in the deterministic concept.

In Table I, the absolute errors of the proposed method are provided in comparison with the absolute errors of LOM using the tau method [28] and the fractional Euler method [14] at $x = 1$. It can be noticed that the proposed method achieves a well approximation with the exact solution only using a few terms of shifted Chebyshev polynomials ($N = 8$). Table I shows that the fractional Euler method [14] has a low accuracy, and it is not a good method for complicated FFDE models. Moreover, a comparison is made between the proposed method and the Leguerre function [28] with the same number of functions. Yet, our method not only has superior accuracy, but is better at the end points of r-cuts as well.

Fig. 2(a) exhibits $\underline{E}_c(1; r)$ for different values of the fuzzy Caputo's fractional derivative. Note that as v approaches 1, the numerical solution converges to the analytical solution of integer-order fuzzy differential equation (i.e., the error decreases gradually). Fig. 2(b) depicts $\underline{E}_c(1; r)$ for the different values of N. As can be seen, with the increasing value of the fractional derivative, the absolute error is decreasing. It is important to note that this behavior does not happen with the growing number of Chebyshev functions, since the proposed method approximates the solution uniformly.

The analytical solution is presented in Fig. 4(a) for $v = 0.85$ defined on the domain $t = [0, 1]$ and $r \in [0, 1]$. The fuzzy approximate solution for (17) is obtained and displayed in Fig. 4(b). As illustrated in Figs. 3 and 4, it is clear that the approximate solution is very accurate at all interval points, specifically near the beginning and end points. Nonetheless, the profiles of Fig. 4(a) and (b) are almost the same.

Although the implementation of [14] is much simpler than our proposed method, it turns out that the fractional Euler method is not a good selection of these type of FDEs, especially when we require higher accuracy. As shown in Table II, it can be noticed that our proposed method can obtain the solution with the functions number $N = 10$, and the CPU time on an Intel (Core i7-3770) 3.40-GHz processor is 7.7790 s and the maximum $\underline{E}_c(1; r)$ is 3.0969e-4. With the same number of Laguerre functions (i.e., $N = 10$), Bhrawy *et al.* [28] could only achieve $\text{Max}(\underline{E}_c(1; r)) =$ 8.6576e-3 and the CPU time is 13.5102 s. Clearly, the latter solution's CPU time is higher than our proposed, while the maximum absolute error is also much lower. This has two important reasons. First, let D_l and D_c be the derivative operational matrices for the Laguerre and Chebyshev functions, respectively. In deriving D_l, only one nonzero term is truncated, but this nonzero term is always equal to -1, independently of the size of D_l. In D_c, only one nonzero term is truncated, but this term diminishes as the size of D_c becomes large, which can affect the global error. Second, the integration interval for the orthogonal calculations and the inner product related to (9) is $[0, \infty)$, which can increase the computation time considerably, especially for the problems that a trigonometric functions is included, while for the Chebyshev polynomials, it is $[0, 1]$.

TABLE II
VISCOELASTICITY: COMPARISON OF THE MAXIMUM ABSOLUTE ERROR VERSUS CPU TIME (SECONDS) WITH $v = 0.85$

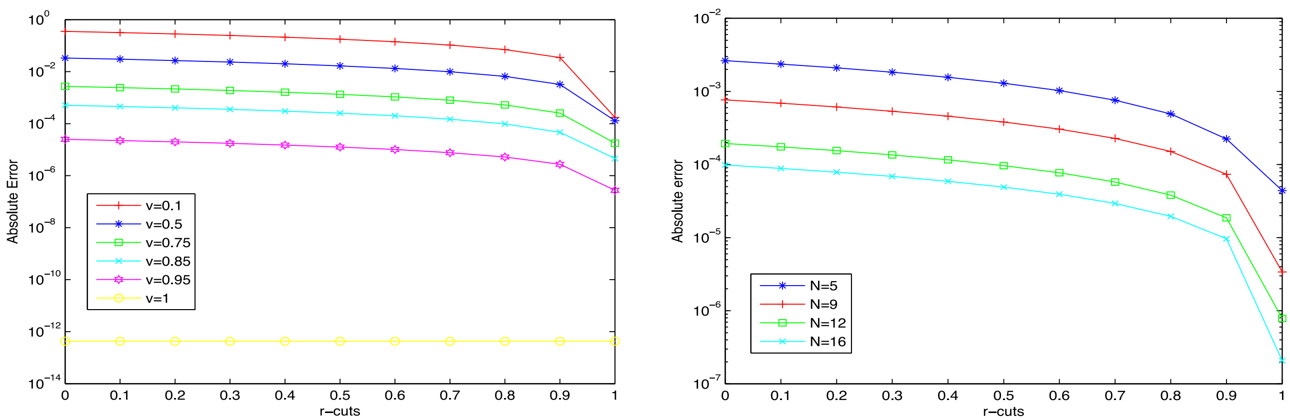
Methods	Max ($\underline{E}_c(1; r)$)	CPU time (Seconds)
Proposed method	3.0969e-4	7.7790
[14]	3.4985e-1	0.0994
[28]	8.6576e-3	13.5102

B. Application II: Oscillation

Harmonic oscillator, given by a linear differential equation of second order with constant coefficients, is a cornerstone of the classical mechanics. Presently, this fundamental conception has the widest origin of physical, chemical, engineering applications and needs no introduction. Its success mainly rests on its

TABLE III
OSCILLATION: COMPARISON OF THE ABSOLUTE ERRORS (E_c, E_l, AND E_u) WITH $v = 0.95$

r	$\underline{Y}_e(1;r)$	$\underline{E}_c(1;r)$	$\underline{E}_l(1;r)$[28]	$\underline{E}_u(1;r)$[14]	$\overline{Y}_e(1;r)$	$\overline{E}_c(1;r)$	$\overline{E}_l(1;r)$[28]	$\overline{E}_u(1;r)$[14]
0	−0.18581	2.15567e-5	2.06735e-3	5.17973e-2	0.55733	2.15150e-5	3.34326e-3	1.99488e-2
0.1	−0.14865	1.94031e-5	1.79682e-3	4.82100e-2	0.52017	1.93614e-5	3.07273e-3	1.63615e-2
0.2	−0.11149	1.72496e-5	1.52629e-3	4.46227e-2	0.48301	1.72078e-5	2.80220e-3	1.27742e-2
0.3	−0.07434	1.50960e-5	1.25576e-3	4.10354e-2	0.44586	1.50542e-5	2.53167e-3	9.18696e-3
0.4	−0.03718	1.29424e-5	9.85233e-4	3.74480e-2	0.40870	1.29006e-5	2.26114e-3	5.59964e-3
0.5	−0.00002	1.07888e-5	7.14702e-4	3.38607e-2	0.37154	1.07470e-5	1.99060e-3	2.01233e-3
0.6	0.03713	8.63524e-6	4.44171e-4	3.02734e-2	0.33438	8.59348e-6	1.72007e-3	1.57497e-3
0.7	0.07428	6.48165e-6	1.73639e-4	2.66861e-2	0.29723	6.43989e-6	1.44954e-3	5.16228e-3
0.8	0.11144	4.32806e-6	9.68912e-5	2.30988e-2	0.26007	4.28629e-6	1.17901e-3	8.74959e-3
0.9	0.14860	2.17447e-6	3.67422e-4	1.95115e-2	0.22291	2.13270e-6	9.08484e-4	1.23369e-2
1	0.18576	2.08802e-8	6.37953e-4	1.59242e-2	0.18576	2.08802e-8	6.37953e-4	1.59242e-2



(a) $N = 8$ (b) $v = 0.85$

Fig. 5. Oscillation: Absolute errors of the proposed method, $\underline{E}_c(1;r)$ (a) for different values of Caputo derivative v and (b) for different values N.

universality, and its simplicity gives boundless intrinsic capabilities for sweeping generalization. Suffice it to recall the passage from the language of functions in phase space to operators in Hilbert space so that the oscillatory model came strongly in the quantum theory. Therefore, the fractional calculus has also made an important contribution to this way. Here, we consider the fractional oscillator to be a generalization of the conventional linear oscillator. Let us consider the fractional oscillation equation with fuzzy initial conditions as

$$\begin{cases} {}^cD^v_{0+}y(x) + y(x) = xe^{-x} \\ y(0,r) = [-1+r, 1-r], \quad 0 < v \leq 1, \quad 0 \leq x \leq 1 \end{cases} \tag{21}$$

where $y \in C(J,\mathbb{E}) \cap L^1(J,\mathbb{E})$ is a continuous fuzzy set-value function, and ${}^cD^v_{0+}$ indicates the fuzzy fractional derivative order of Caputo type.

With respect to Definition 4.6(i) and Theorem 4.1, the parametric form of (21) can be obtained as

$$\begin{cases} ({}^cD^v_{0+}y_-)(x,r) + y_-(x,r) = xe^{-x} \\ y_-(0,r) = -1+r, \quad 0 < v \leq 1, \quad 0 \leq x \leq 1 \end{cases} \tag{22}$$

and

$$\begin{cases} ({}^cD^v_{0+}y_+)(x,r) + y_+(x,r) = xe^{-x} \\ y_+(0,r) = 1-r, \quad 0 < v \leq 1, \quad 0 \leq x \leq 1 \end{cases} \tag{23}$$

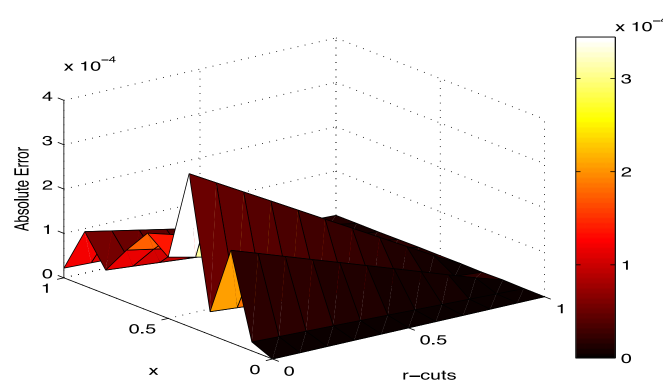
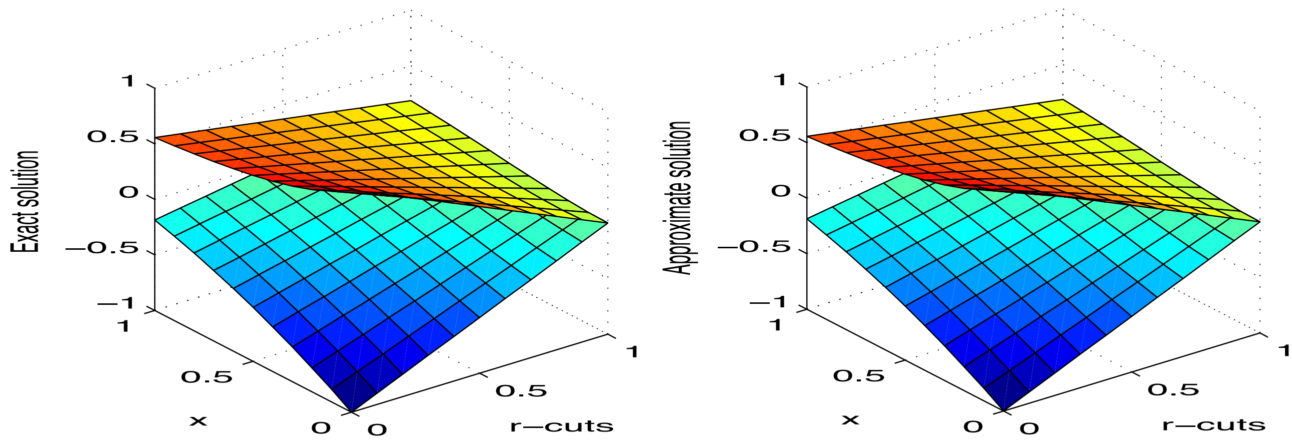


Fig. 6. Oscillation: $\underline{E}_c(t;r)$ for $r \in [0,1]$ and $x \in [0,1]$, $v = 0.95$.

with the exact solution as

$$\begin{cases} y_-(x,r) = (-1+r)E_{v,1}[-x^v] + \int_0^x (x-t)^{v-1} \\ \qquad E_{v,v}[-(x-t)^v]te^{-t}dt, \quad 0 \leq r \leq 1 \\ y_+(x,r) = (1-r)E_{v,1}[-x^v] + \int_0^x (x-t)^{v-1} \\ \qquad E_{v,v}[-(x-t)^v]te^{-t}dt, \quad 0 \leq r \leq 1 \end{cases} \tag{24}$$

where $E_{v,v}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(vr+v)}$ is the generalized Mittag–Leffler function. As stated in Remark 3.1 and (24), it is clear that the analytical solution of (21) is very complicated and hard



(a) (b)

Fig. 7. Profiles of (a) the exact solution and (b) the fuzzy solution of fractional oscillation equation with fuzzy initial conditions with $v = 0.95$ and $N = 12$.

TABLE IV
OSCILLATION: COMPARISON OF THE MAXIMUM ABSOLUTE ERROR VERSUS CPU TIME (SECONDS) WITH $v = 0.85$

Methods	Max ($\underline{E}_c(1;r)$)	CPU time (seconds)
Proposed method	3.1001e-4	4.2987
[14]	4.3741e-2	0.01243
[28]	7.8743e-3	8.7836

TABLE V
RC CIRCUIT: COMPARISON OF THE ABSOLUTE ERRORS (E_c, E_l, AND E_u) WITH $\alpha = 0.98$ AND $N = 10$

r	$\underline{Y}_e(0.008;r)$	$\underline{E}_c(0.008;r)$	$\underline{E}_l(0.008;r)$[28]	$\underline{E}_u(0.008;r)$[14]
0	0.00421	1.55135e-3	5.08708e-3	9.17482e-1
0.1	0.00430	1.55137e-3	5.08403e-3	8.25748e-1
0.2	0.00439	1.55138e-3	5.08098e-3	7.34014e-1
0.3	0.00448	1.55140e-3	5.07794e-3	6.42279e-1
0.4	0.00458	1.55142e-3	5.07489e-3	5.50545e-1
0.5	0.00467	1.55143e-3	5.07184e-3	4.58810e-1
0.6	0.00476	1.55145e-3	5.06879e-3	3.67076e-1
0.7	0.00485	1.55147e-3	5.06575e-3	2.75342e-1
0.8	0.00495	1.55148e-3	5.06270e-3	1.83607e-1
0.9	0.00504	1.55150e-3	5.05965e-3	9.18734e-2
1	0.00513	1.55151e-3	5.05661e-3	1.39073e-4
r	$\overline{Y}_e(0.008;r)$	$\overline{E}_c(0.008;r)$	$\overline{E}_l(0.008;r)$[28]	$\overline{E}_u(0.008;r)$[14]
0	0.00605	1.55168e-3	5.02613e-3	9.17204e-1
0.1	0.00596	1.55166e-3	5.02918e-3	8.25470e-1
0.2	0.00587	1.55165e-3	5.03223e-3	7.33735e-1
0.3	0.00578	1.55163e-3	5.03527e-3	6.42001e-1
0.4	0.00568	1.55161e-3	5.03832e-3	5.50267e-1
0.5	0.00559	1.55160e-3	5.04137e-3	4.58532e-1
0.6	0.00550	1.55158e-3	5.04442e-3	3.66798e-1
0.7	0.00541	1.55156e-3	5.04746e-3	2.75064e-1
0.8	0.00531	1.55155e-3	5.05051e-3	1.83329e-1
0.9	0.00522	1.55153e-3	5.05356e-3	9.15952e-2
1	0.00513	1.55151e-3	5.05661e-3	1.39073e-4

to obtain. Therefore, the numerical simulations with acceptable accuracy are very attractive and highly desirable in this field.

Taking into account (9) and (12), we can reach to the following N fuzzy algebraic linear equation system:

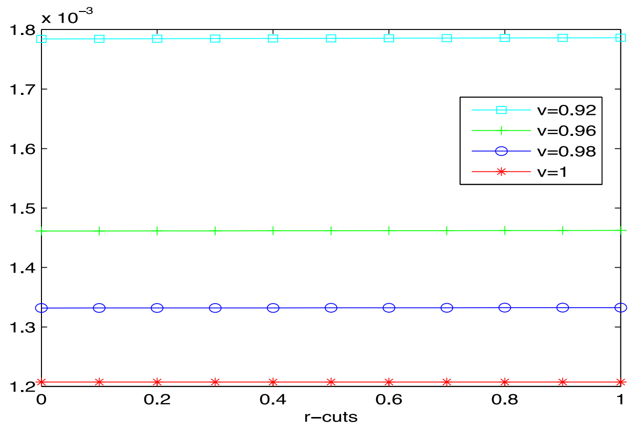
$$\sum_{j=0}^{N-1} c_j(r) \odot \left\{ (FR) \int_0^1 D^{(v)} T_j^*(x) T_i(x) \odot \frac{1}{\sqrt{(x-x^2)}} dx \right.$$
$$\left. + (FR) \int_0^1 T_j^*(x) T_i^*(x) \odot \frac{1}{\sqrt{(x-x^2)}} dx \right\}$$
$$= \sum_{j=0}^{N-1} f_j(r) \odot (FR) \int_0^1 T_j^*(x) T_i^*(x) \odot \frac{1}{\sqrt{(x-x^2)}} dx \quad (25)$$

where $f_i = \frac{1}{h_i} \int_0^1 T_i^*(x) x e^{-x} dx$. Thereafter, replacing (2) in the initial conditions of (21) gives

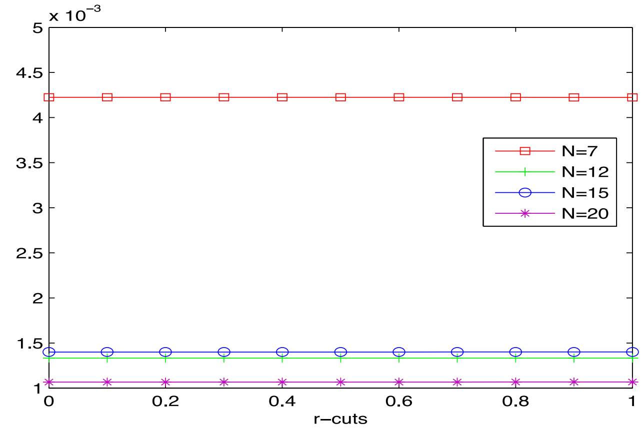
$$y(0,r) = [-1+r, 1-r] = \sum_{j=0}^{N} c_j(r) T_j(0) \quad (26)$$

Equations (25) and (26) give $(N+1)$ fuzzy algebraic linear equations. By solving these equations, we will able to obtain the unknown coefficients of the Chebyshev approximate function of solution.

In Table III, the obtained absolute errors of our method and [14], [28] with $v = 0.95$ and $N = 12$ are reported. This table demonstrates that our proposed method can achieve a higher level of accuracy in comparison with the LOM with the tau method [28] and the fractional Euler method [14]. The proposed method yields better performance than the LOM with the tau method [28] because it offers a more precise solution with the proposed fuzzy method than the LOM with the tau method in the FFDE problem. This is in accordance with our theoretical studies in Section II. The comparison between the proposed method and the LOM with the tau method is based on the same computational complexity (i.e., the same number of functions). Furthermore, Fig. 5(a) depicts $\underline{E}_c(1;r)$ for $v = 0.1, 0.5, 0.75, 0.85, 0.95$, and 1, respectively. Again, this shows that the numerical results from our proposed method are consistent with the exact solutions, and as v approaches 1, the corresponding solutions of (27) approach that of



(a) $N = 12$



(b) $v = 0.98$

Fig. 8. *RC* Circuit: Absolute errors of the proposed method, $\underline{E}_c(0.008; r)$ (a) for different values of Caputo derivative v and (b) for different values N.

integer-order fuzzy differential equation. In addition, Fig. 5(b) illustrates $\underline{E}_c(1; r)$ for a different number of Chebyshev functions. It can be observed that the error rate decreases when N increases. The lower bound of the absolute error function shown in Fig. 6 verifies the reliability of the proposed method in the whole interval. Specifically, the errors tend to approach zero at the beginning and end points. With this, it is confirmed that the proposed method has a smooth approximate solution for the fuzzy fractional oscillation models at the boundary points. The simulation result is shown in Fig. 7, while the profile of the fuzzy exact solution and the proposed fuzzy approximate solution is given in Fig. 7(a) and (b), respectively, with $v = 0.95$ and $N = 12$. Note that the analytical and estimated fuzzy solutions are roughly coincided.

In Table IV, we show the comparison of CPU time. The proposed method obtains better performance than [14], [28] in both of the absolute errors and CPU time. Again, this confirms that the fuzzy approximate solution using the proposed method can achieve high accuracy with low computational time.

Remark 3.2: To make the comparison with the method in [28], we chose the generalized Laguerre parameter ($\alpha > -1$) of Laguerre polynomials, $L_i^{\alpha}(x)$, to be $\alpha = 0$. This value was the best selection for the approximation function.

C. Application III—Resistor–Capacitor Circuit

An *RC* circuit is an electric circuit composed of different combinations of resistors and capacitors driven by a voltage or current source. *RC* circuits are frequent elements in electronic devices and play an important role in the transmission of electrical signals in nerve cells. Importance of this type of circuits is determined with their wide areas of applications: radio receivers, audio systems (e.g., a low-pass audio filter is used to preselect low frequencies before amplification in a subwoofer) and even ac generators. The two most common *RC* circuits employed are high-pass filters and low-pass filters. In this section, we want to depict the numerical behavior of a fractional *RC* circuit under uncertainty. Using the Kirchhoff voltage law, we

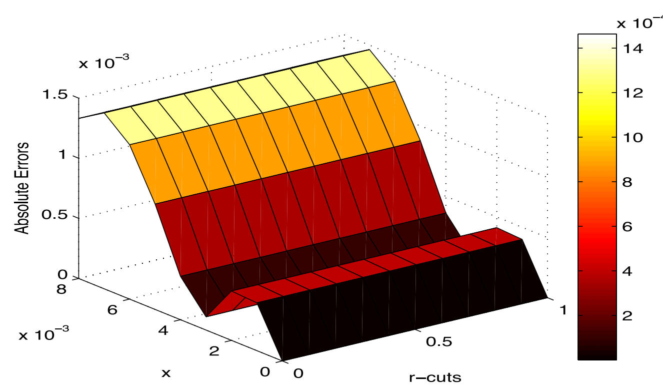


Fig. 9. *RC* Circuit: $\underline{E}_c(t; r)$ for $r \in [0, 1]$ and $t \in [0, 0.008]$, $v = 0.98$, $N = 12$.

have

$$E(t) = V_e(t) + V_C(t) \tag{27}$$

where $E(t)$ is the source voltage, $V_e(t)$ is the voltage in the fractal element, and $V_C(t)$ is the voltage in the capacitor. The voltage in the fractal element is

$$V_e(t) = \frac{e}{\sigma_e^{(1-v)\beta}} \frac{d^{\beta_v} q(t)}{dt^{\beta_v}} \tag{28}$$

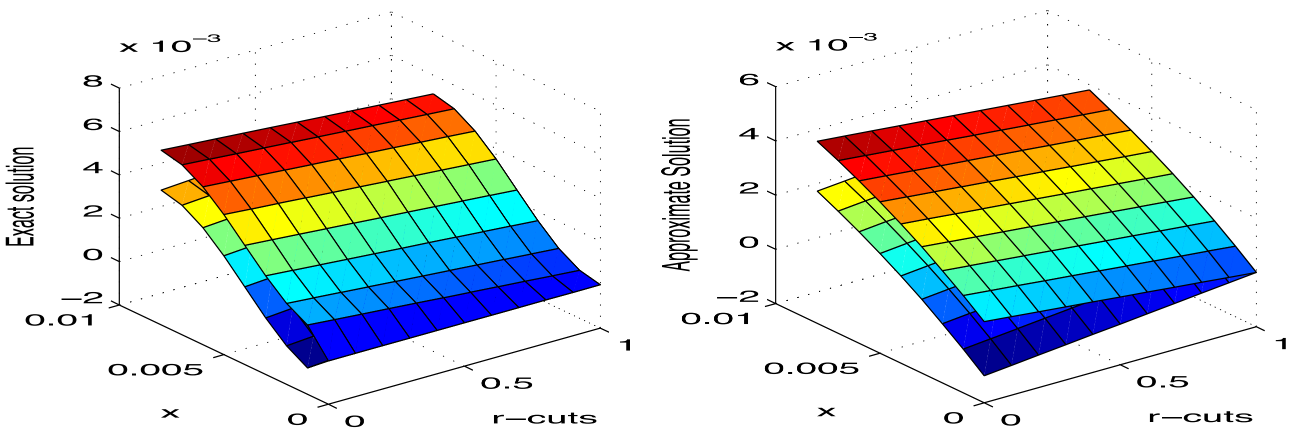
where $\beta\{1, 2\}$ is a parameter that determines whether the element e is a resistor or inductor, $0 < v \leq 1$, the product β_v determines the order of the FDE, $e\{R, L\}$ defines the characteristics of e, and σ is a parameter that determines the fractional structures of e [41].

Assuming in (28) $e = R, \beta = 1,$ and $\sigma_e = \sigma_R$, the FDE under uncertainty for the *RC* circuit has the form:

$${}^cD_{0+}^{v} q(t) + \frac{1}{\tau_v} q(t) = \frac{C}{\tau_v} E(t) \tag{29}$$

where $q(t) \in C(J, \mathbb{E}) \cap L^1(J, \mathbb{E})$ and

$$\tau_v = \frac{RC}{\sigma_R^{1-v}}.$$



(a) (b)

Fig. 10. Profiles of (a) the exact solution and (b) the fuzzy solution of the fractional electrical circuit *RC* under uncertainty, $v = 0.96$ and $N = 12$.

The constant τ_v can also be called fractional time constant due to its dimensionality $[\text{sec}]^v$. When $\tau_v = 1$, (29) recovers the ordinary time constant, i.e., , $\tau_1 = \tau = RC$. The v parameter, which represents the order of the fractional time derivative in (29), can be related to the parameter σ_R, which characterizes the presence of fractional structures (fluctuations) in the system. In this case, the empirical relationship is given by the expression

$$v = \frac{\sigma_R}{RC}.$$

Assuming fuzzy initial conditions, $q(0, r) = [-0.001 + 0.001r, 0.001 - 0.001r]$ and $E(t) = \sin \omega t$.

Now, let $N = 3$ and $v = 0.98$ for (29) and also assume that for the charge, voltage, and current, respectively, using $R = 1$ p.u., $C = 0.1$ p.u., and $\omega = 2\pi 60$. Thus, we can write

$$y_a(t) = \sum_{i=0}^{3} {}^{\bullet} c_i \odot T_i^*(t)$$
$$D^{(0.98)} u_4(t) = \sum_{i=0}^{3} {}^{\bullet} c_i \odot D^{(0.98)} T_i^*(t) \tag{30}$$

where

$$D^{(0.98)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1.9685 & 0.0772 & -0.0375 & 0.0246 \\ -0.0030 & 7.6402 & 0.2024 & -0.1111 \\ 5.6751 & 0.0693 & 11.2898 & 0.3456 \end{bmatrix}.$$

Using (12) and (30), we can write the tau approximation of (29) as

$$\sum_{j=0}^{3} c_j(r) \odot \left\{ \left\langle D^{(0.98)} T_j^*(t), T_i^*(t) \right\rangle + \left\langle T_j^*(t), T_i^*(t) \right\rangle \right\}$$
$$= \sum_{j=0}^{3} f_j(r) \odot \left\langle T_j^*(t), T_i^*(t) \right\rangle \tag{31}$$

TABLE VI
RC CIRCUIT: COMPARISON OF THE MAXIMUM ABSOLUTE ERROR VERSUS CPU TIME (SECONDS) WITH $v = 0.98$

Methods	Max ($\underline{E}_c(0.008; r)$)	CPU time (seconds)
Proposed method	1.1551e-3	2.7584
[14]	9.1654e-1	0.0152
[28]	5.0870e-3	9.9520

for $i = 0, 1, 2$. In addition, the fuzzy coefficients of $f(t) = \frac{C}{\tau_v} \sin \omega t$ are calculated as

$$f(t) = \frac{C}{\tau_v} \sin \omega t \simeq f_4(t) = \sum_{j=0}^{3} f_j T_j^*(t)$$

in which $f_j, j = 0, ..., 3$, are calculated as

$$f_j = \frac{C}{\tau_v h_j} \int_0^1 T_j^*(t) \sin \omega t \frac{1}{\sqrt{t - t^2}} dt.$$

Furthermore, approximate function (2) is substituted in the initial condition, $(t = 0)$ of (29) as

$$y(0, r) = \sum_{j=0}^{3} c_j(r) \odot T_j^*(0) = [-0.001 + 0.001r, 0.001 - 0.001r].$$

Now, we can derive and solve the four fuzzy algebraic linear equations easily to find the unknown coefficients $\{c_j\}_{j=0}^3$. Next, numerical results will be presented for different values of v and N to confirm the validity and efficiency of our proposed method for this specific fuzzy model.

In Table V, the absolute errors between the exact solution $(Y_e(0.008, r))$ and the approximate solution $(y_a(0.008, r))$ at $N = 10$ with the final time $T = 0.008$ are given. The absolute error is compared with the absolute errors of the Laguerre tau spectral method [28] and the fractional Euler method [14], respectively. As usual, the fractional Euler method is the easiest scheme to implement; however, they are of low-order accuracy and has some limitations to provide the numerical solution for

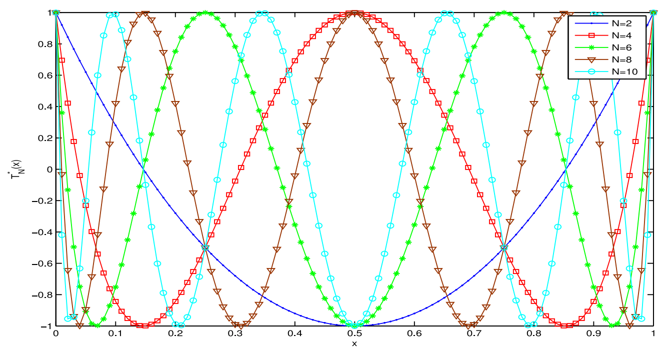


Fig. 11. Graph of the shifted Chebyshev polynomials with various values of N over $[0, 1]$.

a wide range of the fuzzy fractional problems. In contrast, our proposed method offers better accuracy.

The absolute errors ($\underline{E}_c(0.008; r)$) of the system (29) using the tau spectral method based on the COM under uncertainty for different values of v are illustrated in Fig. 8(a). This figure shows that the fuzzy solution converges progressively as v increases from 0.92 to 1. In terms of the physical approach, the fractal element behaves as a resistor of resistance R for $v = 1$, and intermediate values of v determine the behavior between a capacitor and a resistor. In Fig. 8(b), we display absolute errors of various choices of N. It is clear that when N increases, the approximate solution improves. Fig. 9 shows the error profile ($\underline{E}_c(t; r)$) between the exact solution and the proposed fuzzy solution at $t \in [0, 0.008]$ for $v = 0.98$. Again, it is confirmed that the error function has a smooth behavior throughout the reported interval. The fuzzy approximate solution is expressed using the COM with the *tau* method in Fig. 10(b) and compared with the exact solution in Fig. 10(a). Finally, we show the CPU time in Table VI. We observe that although the maximum absolute error of the method in [28] is close to the proposed method, the CPU time is much higher than that in our proposed method.

IV. Conclusion

This paper has presented both numerical simulation and introduced fuzzy mathematical models that can be represented in terms of FDEs under certainty. Compared with the extensive amount of work put into developing FDE schemes in the literature, we found out that only a little effort has been put into developing numerical methods for FFDE. Even so, most of the solutions are based on a rigorous framework, that is, they are often tailored to deal with specific applications and are generally intended for small-scale fuzzy fractional systems. In this paper, we deployed a spectral *tau* method based on Chebyshev functions to reduce the FFDE to a fuzzy algebraic linear equation system to address the fuzzy fractional systems. In fact, in comparison with other methods developed for the FFDE, our scheme has a number of advantages: 1) ease of implementation; 2) lower computational cost; and 3) high accuracy. Although, at the moment, this paper only covers the case of the fuzzy linear time-invariant systems, our future work intends to extend other types of systems such as fuzzy random FDE, fuzzy functional FDE, nonlinear systems, time delay, and time varying using the analogous technique proposed in this paper.

Appendix

In this appendix, necessary definitions and mathematical preliminaries of the fuzzy set theory, fuzzy fractional calculus, and Chebyshev polynomials are revisited.

A. Basic Concepts

Definition 4.1 (see [42]): We define a metric D on $\mathbb{E}$ ($D : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}_+ \bigcup \{0\}$) by a distance, namely, the Hausdorff distance as follows:

$$D(u, v) = \sup_{r\in[0,1]} \max\{|u_-(r) - v_-(r)|, |u_+(r) - v_+(r)|\}. \tag{32}$$

It is shown that $(\mathbb{E}, D)$ is a complete metric space.

The concept of Hukuhara difference, which is recalled in the next definition, was initially generalized by Markov [43] to introduce the notion of generalized Hukuhara differentiability for the interval-valued functions. Afterward, Kaleva [44] employed this notion to define the fuzzy Hukuhara differentiability for the fuzzy-valued functions.

Definition 4.2 (see [42]): Let $x, y \in \mathbb{E}$. If there exists $z \in \mathbb{E}$ such that $x = y \oplus z$, then z is called the Hukuhara difference of x and y, and it is denoted by $x \ominus y$.

In this paper, the sign "$\ominus$" always stands for Hukuhara difference (H-difference for short) and note that $x \ominus y \neq x + (-y)$. In addition, throughout the paper, it is assumed that the H-difference and generalized Hukuhara difference (gH-difference) exist.

Theorem 4.1 (see [45]): Let $F : (a, b) \to \mathbb{E}$. If F is gH-differentiable at $x \in (a, b)$; then, one of the following cases hold:

i) $f'_-(t, r)$, $f'_+(t, r)$ are differentiable at x uniformly in $r \in [0, 1]$, and either

$$[F'(t, r)] = [f'_-(t, r), f'_+(t, r)] \quad \forall r \in [0, 1]$$

or

$$[F'(t, r)] = [f'_+(t, r), f'_-(t, r)] \quad \forall r \in [0, 1].$$

ii) $(f'_-)_1(t, r), (f'_-)_2(t, r), \quad (f'_+)_1(t, r), (f'_+)_2(t, r)$ exist (one-sided derivatives of the end point functions), uniformly in $r \in [0, 1]$, and satisfy $(f'_-)_1(t, r) = (f'_+)_2(t, r)$ and $(f'_-)_2(t, r) = (f'_+)_1(t, r)$ and either

$$\begin{aligned}[F'(t, r)] &= [(f'_-)_2(t, r), (f'_+)_2(t, r)] \\ &= [(f'_+)_1(t, r), (f'_-)_1(t, r)], \quad \forall r \in [0, 1]\end{aligned}$$

or

$$\begin{aligned}[F'(t, r)] &= [(f'_+)_2(t, r), (f'_-)_2(t, r)] \\ &= [(f'_-)_1(t, r), (f'_+)_1(t, r)], \quad \forall r \in [0, 1].\end{aligned}$$

B. Fuzzy Fractional Differentiability

Definition 4.3 (see [46]): We denote the Caputo fractional derivatives by the capital letter with upper-left index ${}^cD$, and the Caputo fractional derivatives of order v are defined as

$$
\begin{aligned}
{}^cD^v f(x) &= I^{m-v} D^m f(x) \\
&= \frac{1}{\Gamma(m-v)} \int_0^x (x-t)^{m-v-1} f^m(t)\, dt
\end{aligned}
$$

where $m-1 < v \le m$, $x > 0$, and D^m is the classical differential operator of order m.

For the Caputo derivative, we have

$$
{}^cD^v C = 0, \quad (C \text{ is a constant})
$$

$$
{}^cD^v x^\beta = \begin{cases} 0, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil v \rceil \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-v)} x^{\beta-v}, & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \ge \lceil v \rceil \\ & \text{or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor v \rfloor . \end{cases}
$$

The ceiling function $\lceil v \rceil$ is used to denote the smallest integer greater than or equal to v, and the floor function $\lfloor v \rfloor$ is to denote the largest integer less than or equal to v. In addition, $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Definition 4.4 (see [47]): Similar to the differential equation of integer order, the Caputo fractional differentiation is a linear operation, i.e.,

$$
{}^cD^v(\lambda f(x) + \mu g(x)) = \lambda {}^cD^v f(x) + \mu {}^cD^v g(x)
$$

where λ and μ are constants.

Let $a > 0$ and $J = (0, a]$; we denote $C(J, \mathbb{E})$ as the space of all continuous fuzzy functions defined on J. Also let $f \in C(J, \mathbb{E})$; we say that $f \in L^1(J, \mathbb{E})$ iff $D(\int_0^a f(s)ds, \hat{0}) < \infty$ [48]. In the rest of the paper, the above notations will be used frequently. The fuzzy Caputo fractional derivatives of order $0 < v \le 1$ for fuzzy-valued function f are given as follows.

Definition 5.5 (see [49]): Let $f \in C(J, \mathbb{E}) \cap L^1(J, \mathbb{E})$ is a fuzzy set-value function; then, f is said to be Caputo's fuzzy differentiable at x when

$$
({}^cD^v_{0^+} f)(x) = \frac{1}{\Gamma(1-v)} \int_0^x \frac{f'(t)}{(x-t)^v} dt \tag{33}
$$

where $0 < v \le 1$.

Definition 4.6 (see [14]): Let $f \in C(J, \mathbb{E}) \cap L^1(J, \mathbb{E})$ and $x_0 \in J$ and $\Phi(x) = \frac{1}{\Gamma(1-v)} \int_0^x \frac{f(t) \ominus \sum_{k=0}^1 \frac{t^k}{k!} f_0^{(k)}}{(x-t)^v} dt$. We say that $f(x)$ is fuzzy Caputo fractional differentiable of order $0 < v \le 1$ at x_0, if there exists an element $({}^cD^v_{0^+} f)(x_0) \in C(J, \mathbb{E})$ such that for all $0 \le r \le 1$ and for $h > 0$ sufficiently near zero on either

$$
i) \qquad ({}^cD^v_{0^+} f)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0 + h) \ominus \Phi(x_0)}{h} = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 - h)}{h}
$$

or

$$
ii) \qquad ({}^cD^v_{0^+} f)(x_0) = \lim_{h \to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0 + h)}{-h} = \lim_{h \to 0^+} \frac{\Phi(x_0 - h) \ominus \Phi(x_0)}{-h}
$$

For the sake of simplicity, we say that the fuzzy-valued function f is ${}^c[(1) - v]$-differentiable if it is differentiable as in Definition 4.6(i), and f is ${}^c[(2) - v]$-differentiable if it is differentiable as in Definition 4.6(ii).

Proposition 4.1 (see [50]): If $\vartheta \in \mathbb{E}$, then we have the following results:

i) $\vartheta(r_2) \subset \vartheta(r_1)$, if $0 < r_1 \le r_2 \le 1$.
ii) $\{r_n\} \subset [0, 1]$ is a nondecreasing sequence which converges to r; then, $\vartheta(r) = \cap_{n \ge 1} \vartheta(r_n)$.

Conversely, if $\mathrm{P}(r) = \{[\mathrm{P}_1(r), \mathrm{P}_2(r)] : r \in [0, 1]\}$ is a family of closed real intervals which (i) and (ii) hold, then $\{\mathrm{P}(r)\}_{r \in [0,1]}$ defines a fuzzy number $\vartheta \in \mathbb{E}$ such that $\mathrm{P}(r) = \vartheta(r)$.

Now, in the following theorem, we prove the fact that under some conditions, the FFDE can be equivalent with an associated fuzzy fractional integral equation. In fact, the following results have been firstly presented and proved in [12] and [19] for interval fractional calculus.

Theorem 4.2: Let $f(x, r) = [f_-(x, r), f_+(x, r)] \in C(J, \mathbb{E}) \cap L^1(J, \mathbb{E})$, f_-, f_+ are Caputo differentiable and $0 < v \le 1$. Then

$$
\begin{aligned}
({}^cD^v_{0^+} f)(x, r) \supseteq [&\min\{({}^cD^v_{0^+} f_-)(x, r), ({}^cD^v_{0^+} f_+)(x, r)\}, \\
&\max\{({}^cD^v_{0^+} f_-)(x, r), ({}^cD^v_{0^+} f_+)(x, r)\}]
\end{aligned} \tag{34}
$$

for a.e. $x \in J$ and $r \in [0, 1]$. In addition

$$
({}^cD^v_{0^+} f)(x, r) = [({}^cD^v_{0^+} f_-)(x, r), ({}^cD^v_{0^+} f_+)(x, r)] \tag{35}
$$

if f is ${}^c[(1) - v]$-differentiable, and

$$
({}^cD^v_{0^+} f)(x, r) = [({}^cD^v_{0^+} f_+)(x, r), ({}^cD^v_{0^+} f_-)(x, r)] \tag{36}
$$

if f is ${}^c[(2) - v]$-differentiable.

Proof: Since f_1 and f_2 are differentiable, we have

$$
f'(x, r) = [\min\{f'_-(x, r), f'_+(x, r)\}, \max\{f'_-(x, r), f'_(x, r)\}].
$$

Then

$$
\begin{aligned}
({}^cD^v_{0^+} f)(x, r) &= \frac{1}{\Gamma(1-v)} \int_0^x (x-s)^{-v} f'(s, r) ds \\
&= \frac{1}{\Gamma(1-v)} \int_0^x (x-s)^{-v} [\min\{f'_-(s, r), f'_+(s, r)\}, \\
&\quad \max\{f'_-(s, r), f'_+(s, r)\}] ds \\
&\supseteq \frac{1}{\Gamma(1-v)} \left[\min\left\{ \int_0^x (x-s)^{-v} f'_-(s, r) ds, \int_0^t (x-s)^{-v} f'_+(s, r) ds \right\}, \right. \\
&\quad \left. \max\left\{ \int_0^x (x-s)^{-v} f'_-(s, r) ds, \int_0^x (x-s)^{-v} f'_+(s, r) ds \right\} \right].
\end{aligned}
$$

In fact

$$
\begin{aligned}
&\frac{1}{\Gamma(1-v)} \int_0^x (x-s)^{-v} \left(\min\left\{ f'_-(s, r), f'_+(s, r) \right\} \right) ds \\
&\le \frac{1}{\Gamma(1-v)} \min\left\{ \int_0^x (x-s)^{-v} f'_-(s, r) ds, \int_0^t (x-s)^{-v} f'_+(s, r) ds \right\} \\
&\le \frac{1}{\Gamma(1-v)} \max\left\{ \int_0^x (x-s)^{-v} f'_-(s, r) ds, \int_0^t (x-s)^{-v} f'_+(s, r) ds \right\} \\
&\le \frac{1}{\Gamma(1-v)} \int_0^x (x-s)^{-v} \max\{f'_-(s, r), f'_+(s, r)\} ds
\end{aligned}
$$

which complete the proof of (34).

Now, suppose that f is ${}^c[(1)-v]$-differentiable, set

$$\mathrm{P}_r := \mathrm{P}(x,r) = \left[\int_0^x (x-s)^{-v} f'_-(s,r)ds, \int_0^x (x-s)^{-v} f'_+(s,r)ds\right].$$

We show that the family $\{\mathrm{P}_r\}_{r\in[0,1]}$ defines a fuzzy-valued function.

Let $r_1 < r_2$; then, $f'_-(s,r_1) \le f'_-(s,r_2)$ and $f'_+(s,r_1) \ge f'_+(s,r_2)$. Hence, $\mathrm{P}(x,r_1) \supseteq \mathrm{P}(x,r_2)$. Using the fact that $f'_-(s,0) \le f'_-(s,r_n) \le f'_-(s,1)$ for $r_n \in [0,1]$, we obtain

$$[|(x-s)^{-v} f'_\dagger(s,r_n)| \le (a)^{-v} \max\{|f'_\dagger(s,0)|, |f'_\dagger(s,1)|\} := h_\dagger(s)$$

for $\dagger = \{-,+\}$. Indeed, $h_\dagger(s)$ is Lebesgue integrable on $(0,a)$. Therefore, using Lebesgue's dominated convergence theorem, if $r_n \overset{n\to\infty}{\to} r$, we have

$$\lim_{n\to\infty}\int_0^x (x-s)^{-v} f'_\dagger(s,r_n)ds = \int_0^x (x-s)^{-v} f'_\dagger(s,r)ds$$

for $\dagger = \{-,+\}$. Consequently, using Proposition 4.1, the proof of 35 is now complete. For Case (36), the idea is completely similar. ■

In the next theorem, we provide and prove the fact that under some conditions the fuzzy Caputo differentiability of the summation of two fuzzy Caputo differentiable functions is linear. This results have been first introduced by Lupulescu [19] for the interval cases. We extended it for some fuzzy cases.

Theorem 4.3: Let $F(x,r) = [f_-(r), f_+(r)], G(x,r) = [g_-(r), g_+(r)] \in C(J,\mathbb{E}) \cap L^1(J,\mathbb{E})$, such that f_-, f_+, g_-, g_+ are Caputo differentiable on J. If F and G are equally fuzzy Caputo differentiable (both are ${}^c[(1)-v]$-differentiable or ${}^c[(2)-v]$-differentiable on J, then

$${}^cD^v_{0^+}(F+G)(x,r) = ({}^cD^v_{0^+}F)(x,r) + ({}^cD^v_{0^+}G)(x,r)$$

for a.e. $x \in J$.

Proof: It is easy to verify that $F+G$ is fuzzy Caputo differentiable. Suppose that F and G are ${}^c[(2)-v]$-differentiable; then

$$\begin{aligned}
{}^cD^v_{0^+}(F+G)(x,r) &= \frac{1}{\Gamma(1-v)}\int_a^x (x-s)^{-v}(F+G)'(s,r)ds \\
&= \frac{1}{\Gamma(1-v)}\int_a^x (x-s)^{-v}\left[f'_+(s,r)+g'_+(s,r), f'_-(s,r)+g'_-(s,r)\right]ds \\
&= \frac{1}{\Gamma(1-v)}\left[\int_a^x (x-s)^{-v}\left(f'_+(s,r)+g'_+(s,r)\right)ds,\right. \\
&\qquad \left.\int_a^x (x-s)^{-v}\left(f'_-(s,r)+g'_-(s,r)\right)ds\right] \\
&= \frac{1}{\Gamma(1-v)}\left\{\int_a^x (x-s)^{-v}\left[f'_+(s,r), f'_-(s,r)\right]ds\right. \\
&\quad + \left.\int_a^x (x-s)^{-v}\left[g'_+(s,r), g'_-(s,r)\right]ds\right\} \\
&= ({}^cD^v_{0^+}F)(x,r) + ({}^cD^v_{0^+}G)(x,r)
\end{aligned}$$

for a.e. $x \in J$. ■

Theorem 4.4: Let $F \in C(J,\mathbb{E}) \cap L^1(J,\mathbb{E})$, and $v \in (0,1]$. If F is ${}^c[(1)-v]$-differentiable, then

$$I^v_{0^+}({}^cD^v_{0^+}F)(x,r) = F(x,r) \ominus F(0,r)$$

or if F is ${}^c[(2)-v]$-differentiable, then

$$I^v_{0^+}({}^cD^v_{0^+}F)(x,r) = -F(0,r) \ominus (-1)F(x,r)$$

for a.e. $x \in J$.

Proof: Let F is ${}^c[(1)-v]$-differentiable; then

$$\begin{aligned}
I^v_{0^+}({}^cD^v_{0^+}F)(x,r) &= I^v_{0^+}I^{1-v}_{0^+}F'(x,r) = I^1_{0^+}F'(x,r) \\
&= \int_0^x F'(s,r)ds = F(x,r) \ominus F(0,r)
\end{aligned}$$

also, if F is ${}^c[(2)-v]$-differentiable, then

$$\begin{aligned}
I^v_{0^+}({}^cD^v_{0^+}F)(x,r) &= I^v_{0^+}I^{1-v}_{0^+}F'(x,r) = I^1_{0^+}F'(x,r) \\
&= \int_0^x F'(s,r)ds = -F(0,r) \ominus (-1)F(x,r)
\end{aligned}$$

which completes the proof. ■

Theorem 4.5: Let $F \in C(J,\mathbb{E}) \cap L^1(J,\mathbb{E})$, and $I^v_{0^+}F$ is ${}^c[(1)-v]$-differentiable; then

$${}^c(I^v_{0^+}F)^{(v)}(x,r) = F(x,r)$$

for a.e. $x \in J$ and $r \in [0,1]$.

Proof: From the assumption, we have $I^v_{0^+}F$ is ${}^c[(1)-v]$-differentiable; therefore, we have

$$\begin{aligned}
&F(x,r) = [f_-(x,r), f_+(x,r)] \\
&\Rightarrow I^v_{0^+}F(x,r) = [I^v_{0^+}f_-(x,r), I^v_{0^+}f_+(x,r)] \\
&\Rightarrow {}^c(I^v_{0^+}F)^{(v)}(x,r) = [{}^c(I^v_{0^+}f_-)^{(v)}(x,r), {}^c(I^v_{0^+}f_+)^{(v)}(x,r)] \\
&\qquad = [f_-(x,r), f_+(x,r)] = F(x,r).
\end{aligned}$$

■

C. Properties of Chebyshev Polynomial

The classical Chebyshev polynomials, denoted by $T_i(x)(i \ge 0)$, as a special case of Jacobi polynomials have been used extensively in mathematical analysis and practical applications as well as play an important role in the analysis and implementation of spectral methods. One of the advantages of using Chebyshev polynomials as a tool for expansion functions is the good representation of smooth functions by finite Chebyshev expansion provided that the function $y(x)$ is differentiable. The coefficients in Chebyshev expansion approach zero faster than any inverse power in n, as n goes to infinity [33]. It is with this motivation that we introduce in this paper a family of Chebyshev polynomials for solving FFDE. They can be determined on the interval $[-1,1]$ with the aid of the following recurrence formula:

$$T_{i+1}(x) = 2tT_i(x) - T_{i-1}(x), \quad i = 1,2,\ldots$$

where $T_0(x) = 1$ and $T_1(x) = t$. In order to use these polynomials on the interval $x \in [0,L]$, we define the so-called shifted Chebyshev polynomials by introducing the change of variable $t = \frac{2x}{L} - 1$. Let the shifted Chebyshev polynomials $T_i(\frac{2x}{L} - 1)$

be denoted by $T^*_{L,i}(x)$ (see Fig. 11). Then, $T^*_{L,i}(x)$ can be obtained as follows:

$$T^*_{L,i+1}(x) = 2\left(\frac{2x}{L} - 1\right) T^*_{L,i}(x) - T^*_{L,i-1}(x), \quad i = 1, 2, \ldots$$

where $T^*_{L,0}(x) = 1$ and $T^*_{L,1}(x) = \frac{2x}{L} - 1$. In this paper, we assume that $L = 1$ and for simplicity denote $T^*_{1,i}(x)$ by $T^*_i(x)$. Therefore, the analytic form of the shifted Chebyshev polynomials $T^*_i(x)$ of degree i is given by

$$T^*_i(x) = i\sum_{k=0}^{i} (-1)^{i-k}\frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!}x^k \tag{37}$$

where $T^*_i(0) = (-1)^i$ and $T^*_i(1) = 1$. The orthogonality condition is

$$\int_0^L T^*_j(x)T^*_k(x)w(x)\,dx = h_k\delta_{\text{jk}} \tag{38}$$

where $w(x) = \frac{1}{\sqrt{x-x^2}}$ and $h_k = \frac{\epsilon_k}{2}\pi, \epsilon_0 = 2, \epsilon_k = 1, k \geq 1$.

A function $f(x)$, square integrable in $[0, 1]$, may be expressed in terms of shifted Chebyshev polynomials as

$$f(x) = \sum_{j=0}^{\infty} c_j T^*_j(x)$$

where the coefficients c_j are given by

$$c_j = \frac{1}{h_j}\int_0^1 f(x)T^*_j(x)w(x)\,dx, \quad j = 0, 1, 2, \ldots. \tag{39}$$

In practice, only the first $(N + 1)$-term shifted Chebyshev polynomials are considered. Therefore, a function $f(x)$ can be expanded as

$$f(x) \simeq f_N(x) = \sum_{j=0}^{N} c_j T^*_j(x) = C^T\Phi(x) \tag{40}$$

where the shifted Chebyshev coefficient vector C and the shifted Chebyshev vector $\Phi(x)$ are given by

$$C^T = [c_0, c_1, \ldots, c_N]$$

$$\Phi(x) = [T^*_0(x), T^*_1(x), \ldots, T^*_N(x)]^T. \tag{41}$$

Suppose that $f_N(x)$ is the best shifted Chebyshev polynomial expansion of $f(x)$ using only the $N + 1$ polynomials $T^*_0(x), \ldots, T^*_N(x)$. We can denote the space generating by these polynomials as

$$\mathbf{T}^N = \text{Span}\{T^*_0(x), T^*_1(x), \ldots, T^*_N(x)\}.$$

Since $\mathbf{T}^N$ is a finite-dimensional vector space, f has a unique best approximation from $\mathbf{T}^N$, say $f_N \in \mathbf{T}^N$, as satisfied in the following inequality [51]:

$$\forall y \in \mathbf{T}^N, \quad \| f(x) - f_N(x) \|_w \leq \| f(x) - y \|_w$$

in which $\| f \|_w = (\int_0^1 f(x)^2 w(x)dx)^{1/2}$ [37].

If we define the q times repeated differentiation of Chebyshev vector $\Phi(x)$ by $D^q\Phi(x)$, then

$$D^q\Phi(x) \simeq D^{(q)}\Phi(x)$$

where q is an integer value and $D^{(q)}$ is the operational matrix of derivative of $\phi(x)$ [31]. Now, in the following theorem, the operational matrix of Caputo fractional derivative of Chebyshev functions is generalized.

Theorem 4.6 (see [24]): Let $\Phi(x)$ is the shifted Chebyshev vector defined in (41) and suppose $v > 0$; then, the Caputo fractional derivatives operator of order v is as

$${}^cD^v\Phi(x) \simeq D^{(v)}\Phi(x) \tag{42}$$

where $D^{(v)}$ is the $(N + 1) \times (N + 1)$ COM of derivative of order v and is defined as follows:

$$D^{(v)} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ S_v(\lceil v\rceil, 0) & S_v(\lceil v\rceil, 1) & S_v(\lceil v\rceil, 0) & \ldots & S_v(\lceil v\rceil, N) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ S_v(i, 0) & S_v(i, 1) & S_v(i, 2) & \ldots & S_v(i, N) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ S_v(N, 0) & S_v(, 1) & S_v(N, 2) & \ldots & S_v(N, N) \end{pmatrix}$$

where

$$S_v(i, j) = \sum_{k=\lceil v\rceil}^{i} \times \frac{(-1)^{i-k}2i(i+k-1)!\Gamma(k-v+\frac{1}{2})}{\epsilon_j\Gamma(k+\frac{1}{2})(i-k)!\Gamma(k-v-j+1)\Gamma(k+j-v+1)}. \tag{43}$$

Note that in $D^{(v)}$, the first $\lceil v\rceil$ rows are all zero.

D. *Proof of Lemma 2.1*

Consider the Taylor's formula

$$y = f(x_0) + f'(x_0)(x - x_0) + \cdots + f^{(N-1)}(x_0)\frac{(x-x_0)^{N-1}}{(N-1)!}$$

for which we know that

$$|f - y| \leq f^{(N)}(\eta)\frac{(x-x_0)^N}{(N)!}, \quad \eta \in (x_0, 1).$$

Since $C^T\Phi(x)$ is the best approximation to f from $\mathbf{T}^N$ and $y \in \mathbf{T}^N$, one has

$$\| f(x) - f_N(x) \|_w^2 \leq \| f - y \|_w^2 \leq \frac{M^2}{(N)!^2}\int_0^1 (x-x_0)^{2(N)}w(x)dx$$

where $M = \max_{x\in[x_0,1]} f^{(N)}(x)$. By choosing $S = \max\{1 - x_0, x_0\}$ and noting that $w(x)$ is always positive in $(0, 1)$, we have

$$\| f(x) - f_N(x) \|_w^2 \leq \frac{M^2S^{2(N)}}{(N)!^2}\int_0^1 w(x)dx = \frac{M^2S^{2(N)}}{(N)!^2}\pi$$

and by taking the square roots, Lemma 2.1 is proved.

E. Proof of Lemma 2.2

First, we obtain a bound for ${}^cD^\alpha x^i$. Therefore, taking Definition 4.3 into consideration, we have

$${}^cD^v x^i = \frac{\Gamma(i+1)}{\Gamma(i+1-v)} x^{i-v} \le \frac{|\Gamma(i+1)|}{|\Gamma(1-v)|} x_0^{-v}, \quad i = 0, 1, \ldots, N.$$

Then, using (37), Definition 4.3, and Chebyshev polynomials properties, we have

$$\begin{aligned} {}^cD^\alpha T_i^*(x) &= i \sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!} {}^cD^\alpha x^k \\ &\le \frac{|\Gamma(i+1)|}{|\Gamma(1-v)|} x_0^{-v} i \sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!} \\ &= \frac{|\Gamma(N+2)|}{|\Gamma(1-v)|} x_0^{-v} T_i^*(1) \end{aligned}$$

but $T_i^*(1) = 1$; therefore, we can obtain

$${}^cD^\alpha T_i^*(x) \le \frac{|\Gamma(i+1)|}{|\Gamma(1-v)|} x_0^{-v}.$$

Now, exploiting Lemma 2.1, Lemma 2.2 is proved.

F. Method of Finding $\{\boldsymbol{c_i}\}_{i=0}^N$

Let us consider again the following linear FDE:

$$({}^cD_{0^+}^v y)(x) + \mu y(x) = f(x), \quad \text{for } 0 < v \le 1, \text{in } \mathbf{I} = (0,1) \tag{44}$$

with initial condition

$$y(0) = y_0 \tag{45}$$

in which μ is a constant. In addition, ${}^cD_{0^+}^v$ indicates the Caputo's fractional derivative of order v for $y(x)$; y_0 describes the initial value of $y(x)$, and $f(x)$ is an arbitrary source function.

Let $w(x) = \frac{1}{\sqrt{x-x^2}}$; then, we denote by $L_w^2(\mathbf{I})(\mathbf{I} := (0,1))$ the weighted L^2 space with inner product:

$$\langle u, v\rangle_w = \int_{\mathbf{I}} w(x)u(x)v(x)dx \tag{46}$$

and the associated norm $\| u \|_w = \langle u, u\rangle_w^{1/2}$. It is known that $\{T_n^* : n \ge 0\}$ forms a complete orthogonal system in $L_w^2(\mathbf{I})$. Hence, as stated already, if we define

$$T^N(\mathbf{I}) = \{T_0^*(x), T_1^*(x), \ldots, T_N^*(x)\} \tag{47}$$

then the shifted Chebyshev tau approximation to (44) is to find $y_N \in T^N(\mathbf{I})$ such that

$$\begin{aligned} &\left\langle D^{(v)} y_N, T_k^*(x)\right\rangle_w + \mu\langle y_N, T_k^*(x)\rangle_w = \langle f, T_k^*(x)\rangle_w \\ &k = 0, 1, \ldots, N-1. \end{aligned} \tag{48}$$

As we know

$$\begin{aligned} y_N(x) &= \sum_{j=0}^{N} \boldsymbol{c}_j T_j^*(x), \quad \boldsymbol{C} = [\boldsymbol{c_0}, \boldsymbol{c_1}, \ldots, \boldsymbol{c_N}]^T \\ f_k &= (\langle f, T_k^*(x)\rangle_w, \quad k = 0, 1, \ldots N-1 \\ \boldsymbol{f} &= (\boldsymbol{f_0}, \boldsymbol{f_1}, \ldots, \boldsymbol{f_N}, \boldsymbol{y_0})^T \\ y(0) &= \boldsymbol{y_0}. \end{aligned} \tag{49}$$

Then, (48) can be written as

$$\begin{aligned} &\sum_{j=0}^{N} \boldsymbol{c_j} \left[\left\langle D^{(v)} T_j^*(x), T_k^*(x)\right\rangle_w + \mu\langle T_j^*(x), T_k^*(x)\rangle_w\right] \\ &= \langle \boldsymbol{f}, T_k^*(x)\rangle_w, \quad k = 0, 1, \ldots, N-1, \\ &\sum_{j=0}^{N} \boldsymbol{c_j} T_j^*(0) = \boldsymbol{y_0}. \end{aligned} \tag{50}$$

Let us denote

$$\mathfrak{A} = (a_{kj})_{0<k,j<N}, \quad \mathfrak{B} = (b_{kj})_{0<k,j<N}$$

where

$$a_{kj} = \begin{cases} \left\langle D^{(v)} T_j^*(x), T_k^*(x)\right\rangle_w, & k = 0, 1, \ldots N-1, \ j = 0, 1, \ldots N \\ T_j^*(0), & k = 0, 1, \ldots N-1, \ j = 0, 1, \ldots N \end{cases}$$

$$b_{kj} = \begin{cases} \left\langle T_j^*(x), T_k^*(x)\right\rangle_w, & k = 0, 1, \ldots N-1, \ j = 0, 1, \ldots N \\ 0, & \text{otherwise.} \end{cases}$$

Then, by taking into consideration (43) and employing of the orthogonality relation of shifted Chebyshev polynomials (38), and after some manipulation and calculation, one can depict that the nonzero elements of a_{kj} and b_{kj} are given clearly in the following form:

$$a_{kj} = \begin{cases} h_k S_v(j,k), \\ \qquad k = 0, 1, \ldots N-1, \ j = 0, 1, \ldots N \\ (-1)^{j-k+N+1} \dfrac{j(j-k+N+2)2^{(k-N)} 3\sqrt{\pi}}{\Gamma(k-N-\frac{1}{2})} \\ \qquad k = 0, 1, \ldots N-1, \ j = 0, 1, \ldots N \end{cases}$$

$$b_{kj} = \begin{cases} h_k, & k = j = 0, 1, \ldots N-1 \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we can write (50) in the following matrix system form:

$$(\mathfrak{A} + \mu\mathfrak{B})\boldsymbol{C} = \boldsymbol{f}. \tag{51}$$

In the case of $\mu \ne 0$, the linear system (51) can be solved by employing any existing direct or numerical method.

G. Inner Product

Definition 4.7 (see [52]): An *inner product* on S is a map

$$\begin{aligned} \langle ., . \rangle &: S \times S \to X \\ (p, q) &\mapsto \langle p, q\rangle \end{aligned}$$

where S is a finite-dimensional nonzero vector space over X, with the following properties.

1) *Linearity:* For $p, q, m \in S$

$$\begin{cases} \langle p+m, q\rangle = \langle p, q\rangle + \langle m, q\rangle \\ \langle ap, q\rangle = a\,\langle p, q\rangle\,. \end{cases}$$

2) *Positivity:* $\langle p, p\rangle \geq 0$, for all $p \in S$.
3) *Positive definiteness:* $\langle p, p\rangle = 0$ iff $p = 0$.
4) *Conjugate symmetry:* $\langle p, q\rangle = \overline{\langle q, p\rangle}$ for all $p, q \in S$.

Lemma 4.1: $\langle R_N(x), T_i^*(x)\rangle_{\mathbb{E}}$ is a *fuzzy-like residual inner product* over $X_{\mathbb{E}} = L^2(J, \mathbb{E})$, where

$$\langle R_N(x), T_i^*(x)\rangle_E = (FR)\int_0^1 (R_N(x)\odot T_i^*(x)\odot w(x))dx. \tag{52}$$

Proof: Indeed, $R_N(x,r) = [\underline{R}_N(x,r), \overline{R}_N(x,r)]$, where $\underline{R}_N(x,r)$ and $\overline{R}_N(x,r)$ are given as

$$\begin{aligned}\underline{R}_N(x,r) &= \underline{C}^T(r)(D^{(v)}\Phi(x)+\Phi(x)) - \underline{F}^T(r)\Phi(x)\\ \overline{R}_N(x,r) &= \overline{C}^T(r)(D^{(v)}\Phi(x)+\Phi(x)) - \overline{F}^T(r)\Phi(x).\end{aligned}$$

Therefore, the r-cut representation of (52) is as follows:

$$\langle R_N(x,r), T_i^*(x)\rangle_E = (FR)\int_0^1 (R_N(x,r)\odot T_i^*(x)\odot w(x))\,dx$$

$$= \left[\int_{P_1}\underline{R}_N(x,r)T_i^*(x)w(x)dx + \int_{P_2}\overline{R}_N(x,r)T_i^*(x)w(x)dx,\right.$$

$$\left.\int_{P_1}\overline{R}_N(x,r)T_i^*(x)w(x)dx + \int_{P_2}\underline{R}_N(x,r)T_i^*(x)w(x)dx\right]$$

where P_1 and P_2 are the set of all points that $T_i^*(x) \geq 0$ and $T_i^*(x) < 0$, respectively. Then, it is easy to verify that

$$\int_{P_1}\underline{R}_N(x,r)T_i^*(x)w(x)dx + \int_{P_2}\overline{R}_N(x,r)T_i^*(x)w(x)dx$$

and

$$\int_{P_1}\overline{R}_N(x,r)T_i^*(x)w(x)dx + \int_{P_2}\underline{R}_N(x,r)T_i^*(x)w(x)dx$$

are both inner product over $X_{\mathbb{R}}$. In fact, these relations satisfy in the inner product properties 1–4 in Definition (4.7). ■

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Ali Ahmadian (M’14) received the Ph.D. degree in applied mathematics from the Universiti Putra Malaysia, Serdang, Selangor, Malaysia, in 2013.

He joined the Centre of Image and Signal Processing, Faculty of Computer Science and Information Technology, University of Malaya, Kuala Lumpur, Malaysia, as a Researcher of fuzzy analysis in 2014. He has published more than 30 peer-reviewed scientific publications, including three book chapters and 20 JCR-SCI-indexed journal papers. He is a reviewer for 20 international journals. He was involved in several national projects related to the applications of fuzzy systems in the real-world systems. His current research interests include fuzzy fractional calculus, interval-valued functions, spectral methods for the solution of fuzzy differential equations, and fuzzy mathematical modeling.

Soheil Salahshour received the Ph.D. degree from the Islamic Azad University (IAU), Science and Research Branch, Tehran, Iran, in 2012.

He joined the Department of Mathematics, IAU, Mobarakeh branch, Mobarakeh, Iran, in 2010, where he is currently an Assistant Professor. He has worked in various fields of fuzzy setting theory including fuzzy fractional calculus, fuzzy expert systems, and fuzzy system of equations. He published more than 100 papers in the peer-reviewed journals. He is a reviewer of the several reputed JCR-indexed journals in fuzzy field.

Chee Seng Chan (S’05–M’09–SM’14) received the Ph.D. degree from the University of Portsmouth, Portsmouth, U.K., in 2008.

He is currently a Senior Lecturer with the Faculty of Computer Science and Information Technology, University of Malaya, Kuala Lumpur, Malaysia. In general, his major research interests include computer vision and fuzzy qualitative reasoning, with a focus on image/video content analysis and human–robot interaction.

Dr. Chan is the Founding Chair of the IEEE Computational Intelligence Society Malaysia chapter and the Founder of the Malaysian Image Analysis and Machine Intelligence Association. He is a recipient of the Young Scientist Network-Academy of Sciences Malaysia (YSN-ASM) in 2015, the Hitachi Research Fellowship in 2013, and the Institution of Engineering and Technology (Malaysia) Young Engineer Award in 2010. He is also a Chartered Engineer and a Member of the Institution of Engineering and Technology.