Logical Connectives and Operativeness of BK Sub-triangle Product in Fuzzy Inferencing

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Abstract

Bandler and Kohout's sub-triangle product is well-known due to its ability to retrieve relations of elements in the sets which are not directly associated. In practice, the sub-triangle product is able to generate a list of inference structures that can work as inference engine. In this paper, we examine this sub-triangle product in a medical expert system where lists of equations were initialized from the sub-triangle product as the inference structures. Two limitations were discovered where the former arises from the ignorance of non-emptiness condition, whereas the latter arises from the initialization of an inappropriate logical connectives. To rectify this problem, we proposed to eliminate those inference structures that are not performing well. With a further study on the behaviour of well performing inference structures, we first proposed a combination of inference structures that good for forming inference engines of medical expert systems.

Keywords: BK sub-triangle product, fuzzy relation, inference structure, logical connective.

1. Introduction

Relations between two indirectly associated sets can be studied with the relational products proposed by Bandler and Kohout [1]. These relational products are widely known as BK relational products in the literature. Distinct from the traditional composition of relations [2], BK relational products define the relations between elements within 2 indirectly associated sets as the overlapping of their images in a common set.

In term of applications, BK relational products gained remarkable successful in developing inference engines for numerous applications, such as medical expert systems [3], information retrieval [4], path finding of autonomous underwater vehicles [5], land evaluation [6] and etc. Among many types of fuzzy relations, Kerre [7] described BK relational products as "the most important operation on relations". There are 3 types of relational products defined by Bandler and Kohout, and the BK sub-triangle product is the most used in developing fuzzy inference structures. Moreover, recently Stepnicka and Jayaram [8] also proved that BK sub-triangle product has its noteworthy advantages over compositional rule of inference [9], which is relatively popular.

In order to develop fuzzy inference structures with BK sub-triangle products, appropriate logical connectives must be defined to combine relevant terms. Yew [10] and Yew and Kohout [3, 11-12] showed a typical example of this work where a set of 23 inference structures based on BK sub-triangle product and its variants were developed. These inference structures were tested as an inference engine of a medical expert system, and the performance of each inference structures is recorded and compared. From here, we found out that there are two limitations in formulating of the inference structures which are not addressed. These limitations involve initializing inappropriate logical connectives for inference structures and ignorance of non-emptiness condition.

In this paper, our contributions are the discovery of two limitations presented in the list of inference structure that proposed by Yew and Kohout [3, 11-12]. Besides, we studied the influence of additional term proposed by De Baets and Kerre [13]. With the understanding of the shortcomings and the influence of additional term, we rejected some of the inference structures and minimize the list from 19 to 4. With the combinations of these 4 inference structures, we can form robust inference engines.

The rest of this paper is organized as follows: Section 2 revisits the definition of BK relational products as well as the implementation of the sub-triangle product in [3, 10]. We explained the limitations of their work in Section 3. The influence of the additional term by De Baets and Kerre was studied in Section 4. The proposed method and experiment on simulated data is presented in Section 5. Lastly, we draw conclusion in Section 6.

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Manuscript received Aug. 2011; revised Dec. 2011; accepted Dec. 2011.

2. Revisits the BK Relational Products

A. *BK relational products*

We start the discussion with a brief revision on the fundamental definitions of BK (crisp) relational products. To make the discussion more concise, we use the following notations for the explanation of these definitions, as well as the remaining of this paper.

Set $A=\{a_i \mid i=1,\cdots,m\}$ and set $B=\{b_i \mid i=1,\cdots,n\}$. R is defined as a relation from A to B such that $R\subseteq A\times B$. The abbreviation *aRb* shows that *a* is in relation *R* with *b*.

Definition 1 (Domain): Domain of a relation *R* is a set of elements in *A* such that these elements have relation *R* with at least 1 element in *B*:

$$\mathrm{dom}(R)=\{a \mid a\in A \text{ and } (\exists b\in B)(aRb)\} \quad (1)$$

Definition 2 (Range): Range of a relation *R* is a set of elements in *B* such that these elements can be related by at least 1 element in *A* through relation *R*:

$$\mathrm{rng}(R)=\{b \mid b\in B \text{ and } (\exists a\in A)(aRb)\} \quad (2)$$

Definition 3 (Converse Relation): Converse relation R^T is the reverse of relation *R* from *B* to *A*:

$$R^T=\{(b,a) \mid (b,a)\in B\times A \text{ and } aRb\} \quad (3)$$

Definition 4 (Afterset): Afterset *aR* is the images of *a* under relation *R* in *B*:

$$aR=\{b \mid b\in B \text{ and } aRb\} \quad (4)$$

Definition 5 (Foreset): Foreset *Rb* is the set of *a* which can be related to the particular *b* with relation *R*:

$$Rb=\{a \mid a\in A \text{ and } aRb\} \quad (5)$$

Assume that is another crisp relation *S*, from set *B* to set *C*. These relational products can be defined as follow:

Definition 6 (Sub-triangle product): Sub-triangle product shows all *(a, c)* couples for which the afterset *aR* is a subset of foreset *Sc*

$$R\triangleleft S=\{(a,c) \mid (a,c)\in A\times C \text{ and } aR\subseteq Sc\} \quad (6)$$

Definition 7 (Super-triangle product): Super-triangle product shows all *(a, c)* couples for which the foreset *Sc* is a subset of afterset *aR*:

$$R\triangleright S=\{(a,c) \mid (a,c)\in A\times C \text{ and } Sc\subseteq aR\} \quad (7)$$

Definition 8 (Square product): Square product shows all *(a, c)* couples for which the afterset a*R* is exactly equal to the foreset *Sc*:

$$R\Diamond S=\{(a,c) \mid (a,c)\in A\times C \text{ and } aR=Sc\} \quad (8)$$

As one can notice, the central thought of BK relational products is subsethood. Therefore, these crisp relational products can be developed to fuzzy relational products easily by introducing a fuzzy subsethood (or similarity) measurement. For example, the BK fuzzy sub-triangle product can be expressed as follow:

$$R\triangleleft S(a,c)=\bigwedge_{b\in B}(R_{ab}\rightarrow S_{bc}) \quad (9)$$

Here, $\rightarrow$ represents fuzzy implication operators, R_{ab} is the short form of membership function of *R(a,b)* and S_{bc} is the short form of membership function of *S(b,c)*.

B. *Improvement by De Baets and Kerre*

De Baets and Kerre [13] pointed out that there is a blind side in the definition of BK sub-triangle product, i.e. an element \$a\$ can have relation $R\triangleleft S$ with all the elements in *C* even if there is no image of *a* in *B* under relation *R*. For each BK super-triangle product and square product, a similar imperfection holds. Thus, De Baets and Kerre conclude that a lot of unwanted couples may be generated by the traditional BK relational products.

To rectify this imperfection, De Baets and Kerre [13] proposed that a non-emptiness condition should be added to (6), so that:

$$R\triangleleft^* S=\{(a,c) \mid (a,c)\in A\times C \text{ and } \varnothing\subset aR\subseteq Sc\} \quad (10)$$

The superscripted * for the sub-triangle indicate a product. To fuzzify (10), two equivalent expressions were developed and each lead to a different fuzzy expression.

The first expression is based on the Cartesian product of the domain of the first relation and the range of the second relation:

$$R\triangleleft^* S=(R\triangleleft S\cap(\mathrm{dom}(R)\times\mathrm{rng}(S)) \quad (11)$$

and this expression leads to the first set of improved fuzzy sub-triangle product:

$$R\triangleleft_b S(a,c)=\min(\bigwedge_{b\in B}(R_{ab}\rightarrow S_{bc}),\bigwedge_{b\in B}R_{ab},\bigwedge_{b\in B}S_{bc}) \quad (12)$$

The second expression was developed by intersecting the BK sub-triangle product and classical composition of relations \cite{Zadeh1965}:

$$R\triangleleft^* S=(R\triangleleft S)\cap(R\circ S) \quad (13)$$

and this expression leads to the second set of improved fuzzy sub-triangle product:

$$R\triangleleft_k S(a,c)=\min(\bigwedge_{b\in B}(R_{ab}\rightarrow S_{bc}),\bigwedge_{b\in B}\tau(R_{ab},S_{bc})) \quad (14)$$

The subscripts *b* and *k* in (12) and (14) indicate 2 different fuzzy expressions derived by De Baets and Kerre.

One compared (12) and (14) to the original (9), it is easy to notify that the original expression was expended with additional term(s). We refer the original term that originate from Bandler and Kohout as **Implication term**; whereas the term(s) added by De Baets and Kerre [13] as **Additional term**.

C. *BK Sub-triangle Product in Medical Expert System by Yew and Kohout*

In [3, 10] Yew and Kohout acquired all the original and improved versions of fuzzy sub-triangle product to form 3 fuzzy inference templates. With these inference templates, a set of 23 inference structures as illustrated in Figure 1 were formed and tested in a medical expert system.

Practically, in the medical expert system, (9), (12) and (14) were developed to become Sub-BK inference template, Sub-B inference template and Sub-K inference template, respectively:

Sub-BK Inference Template:

$$(R \lhd S)_{ac} = \curlywedge_2 (R_{ab} \to S_{bc}) \tag{15}$$

Sub-B Inference Template:

$$(R \lhd_b S)_{ac} = \curlywedge_1 (\curlywedge_2 (R_{ab} \to S_{bc}), \curlyvee_3 R_{ab}, \curlyvee_4 S_{bc}) \tag{16}$$

Sub-K Inference Template:

$$(R \lhd_k S)_{ac} = \curlywedge_1 (\curlywedge_2 (R_{ab} \to S_{bc}), \curlyvee_3 (\curlywedge_4 (R_{ab}, S_{bc}))) \tag{17}$$

where all the $\curlywedge_i (i = \{1,2,3\})$ and $\curlyvee_j (j = \{2,3\})$ are logical connectives that yet to instantiate. With checklist paradigm [1, 14], Yew and Kohout found that the following logical connectives are suitable candidates for $\curlywedge_i$ and $\curlyvee_j$:

$$\begin{aligned} &\curlywedge_1 = \{\min, \max\} \\ &\curlywedge_2 = \{\text{Arithemetic mean, AndTop, AndBot}\} \\ &\curlyvee_3 = \{\text{Arithemetic mean, OrTop, OrBot}\} \\ &\curlyvee_4 = \{\text{OrTop}\} \\ &\curlywedge_4 = \{\text{AndTop, AndBot}\} \end{aligned} \tag{18}$$

AndTop, *AndBot*, *OrTop* and *OrBot* are logical connectives derived from checklist paradigm and are defined as follow:

$$\text{AndTop}(p,q) = \min(p,q) \tag{19}$$

$$\text{AndBot}(p,q) = \max(0, p+q-1) \tag{20}$$

$$\text{OrTop}(p,q) = \max(p,q) \tag{21}$$

$$\text{OrBot}(p,q) = \min(1, p+q) \tag{22}$$

Therefore, with (9) and (18), a list of 3 Sub-BK inference structures was instantiated.

$$\text{BK1} = \frac{1}{N} \sum_{b=1}^{N} (R_{ab} \to S_{bc})$$

$$\text{BK2} = \text{AndTop}(R_{ab} \to S_{bc})$$

$$\text{BK3} = \text{AndBot}(R_{ab} \to S_{bc})$$

There is only 1 instantiation for Sub-B inference structure:

$$B_1 = \min(\frac{1}{N} \sum_{b=1}^{N} (R_{ab} \to S_{bc}), \max R_{ab}, \max S_{bc})$$

For Sub-K inference template, a list of 19 inference structures as illustrated in Figure 1 was instantiated. This set of inference structures is the focus of discussion in this paper. In addition, Łukasiewicz and Kleene-Dienes implication operators that defined as follow were chosen as implication operators for all the inference structure.

$$\begin{aligned}
K_1 &= \min(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \frac{1}{N}\sum_{b=1}^{N}(\text{AndTop}(R_{ab}, S_{bc}))) \\
K_2 &= \min(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \frac{1}{N}\sum_{b=1}^{N}(\text{AndBot}(R_{ab}, S_{bc}))) \\
K_3 &= \max(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \frac{1}{N}\sum_{b=1}^{N}(\text{AndTop}(R_{ab}, S_{bc}))) \\
K_4 &= \max(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \frac{1}{N}\sum_{b=1}^{N}(\text{AndBot}(R_{ab}, S_{bc}))) \\
K_5 &= \max(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \text{OrBot}(\text{AndTop}(R_{ab}, S_{bc}))) \\
K_6 &= \max(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \text{OrBot}(\text{AndBot}(R_{ab}, S_{bc}))) \\
K_7 &= \min(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \text{OrBot}(\text{AndBot}(R_{ab}, S_{bc}))) \\
K_8 &= \max(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \text{OrTop}(\text{AndTop}(R_{ab}, S_{bc}))) \\
K_9 &= \min(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \text{OrBot}(\text{AndTop}(R_{ab}, S_{bc}))) \\
K_{10} &= \min(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \text{OrTop}(\text{AndBot}(R_{ab}, S_{bc}))) \\
K_{11} &= \max(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \text{OrBot}(\text{AndBot}(R_{ab}, S_{bc}))) \\
K_{12} &= \min(\frac{1}{N}\sum_{b=1}^{N}(R_{ab} \to S_{bc}), \text{OrTop}(\text{AndTop}(R_{ab}, S_{bc}))) \\
K_{13} &= \min(\text{AndTop}(R_{ij} \to S_{jk}), \text{OrTop}(\text{AndTop}(R_{ij}, S_{jk}))) \\
K_{14} &= \min(\text{AndTop}(R_{ij} \to S_{jk}), \text{OrTop}(\text{AndBot}(R_{ij}, S_{jk}))) \\
K_{15} &= \min(\text{AndBot}(R_{ij} \to S_{jk}), \text{OrBot}(\text{AndBot}(R_{ij}, S_{jk}))) \\
K_{16} &= \min(\text{AndBot}(R_{ij} \to S_{jk}), \text{OrBot}(\text{AndTop}(R_{ij}, S_{jk}))) \\
K_{17} &= \max(\text{AndTop}(R_{ij} \to S_{jk}), \text{OrBot}(\text{AndTop}(R_{ij}, S_{jk}))) \\
K_{18} &= \min(\text{AndTop}(R_{ij} \to S_{jk}), \text{OrBot}(\text{AndTop}(R_{ij}, S_{jk}))) \\
K_{19} &= \min(\text{AndTop}(R_{ij} \to S_{jk}), \text{OrBot}(\text{AndBot}(R_{ij}, S_{jk})))
\end{aligned}$$

Figure 1. List of Sub-K Inference Structures.

- Łukasiewicz implication operator, $p \to_{Ł}$

$$p \to_{Ł} q = \min(1, 1-p+q) \tag{23}$$

Kleene-Dienes implication operator, $p \to_{KD}$

$$p \to_{KD} q = \max(1-p+q) \tag{24}$$

With this set of implication operators, each inference yield an interval in the range [0,1]. The upper bound of an inference is given by $\to_{Ł}$ whereas the lower bound is given by $p \to_{KD}$.

To evaluate the inference structures, the resulted intervals of inferences are considered as accepted if they fall into a predefined accepted threshold (e.g: higher then 0.8 considered accepted). On the other hand, an inference is rejected if the interval fall into a predefined rejected threshold (e.g: lower then 0.3 considered rejected).

If an inference not able to accept the true disease, the true acceptance rate is 0; otherwise the true acceptance rate is given by *1/number of accepted diseases*. It is an instance of false rejection if the true disease is rejected. On the other hand, false acceptance rate is the proportion of the incorrect diseases being accepted. The proportion of the incorrect diseases being rejected is true rejection rate. [3,10] evaluated the performance of the inference structures based on their mean values of inference results, i.e. mean true acceptance (MTA), mean true rejection (MTR), mean false acceptance (MFA) and mean false rejection (MFR). Table 1 showed the performance results of all the inference structures tested in the system. We can noticed that the results are not convincing as a good inference structures should show high MTA and MTR, and low MFA and MFR.

3. Limitations of Yew's Application of BK Sub-triangle Product

A. *BK relational products*

As discussed earlier, in [3,10], a list of 19 inference structures based on *K* inference templates was proposed. Among these 19 inference structures, 7 of them employed *max* as the outer logical connective, $\curlywedge_1$. These inference structures are K3, K4, K5, K6, K8, K11 and K17.

Table 1. Ranking of Inference Structures at Threshold of Acceptance 0.8, Threshold of Rejection 0.3, Sorted According to MTA.

Inference Structures	MTA	MTR	MFA	MFR
K7	0.70	0.33	0.15	0.00
BK2	0.61	0.22	0.04	0.00
K9	0.54	0.11	0.33	0.00
BK3,K19	0.50	0.48	0.04	0.00
K18	0.50	0.26	0.04	0.00
K17	0.48	0.07	0.56	0.00
K12	0.44	0.33	0.07	0.00
BK1	0.44	0.00	0.67	0.00
K3,K4	0.43	0.00	0.74	0.00
K15	0.39	0.59	0.04	0.00
K16	0.39	0.52	0.04	0.00
B1	0.39	0.33	0.11	0.00
K6	0.37	0.00	0.78	0.00
K10	0.33	0.52	0.07	0.00
K8	0.31	0.00	0.81	0.00
K14	0.28	0.59	0.04	0.00
K13	0.28	0.41	0.04	0.00
K11	0.27	0.00	0.93	0.00
K5	0.26	0.00	0.96	0.00
K2	0.00	0.96	0.00	0.67
K1	0.00	0.63	0.00	0.67

Please note that the purpose of $\curlywedge_1$ in Sub-K Inference Template is to pick a candidate from two to be the result of the particular inference [refer to (17)]. The first candidate is the implication term that proposed by the original BK Sub-triangle product $\curlywedge_2(R_{ab} \to S_{bc})$, whereas the second term is the additional term that proposed by De Baets and Kerre \cite{DeBaets1993} for fulfilling non-emptiness condition $\curlyvee_3(\curlywedge_4(R_{ab}, S_{bc}))$. In case of the emptiness present, operations on implication term most probably yield a large number due to the nature of implication operators $p \to q$ if *p=0*.

Obviously, using a *min* as the $\curlywedge_1$ can cause the result of the particular inference come from the additional term. Hence, the problem of emptiness solved with the solution proposed by De Baets and Kerre [13]. However, using *max* as $\curlywedge_1$ will take the implication term as output of inference, which implies that the non-emptiness condition is ignored again.

So, it is clear that *max* is not a valid outer connective in solving the Sub-K Inference Template due to the non-emptiness condition. The same conclusion hold if Sub-B Inference Template is studied and the reason is trivial.

B. *Using AndBot to Aggregate the Results of Implications*

In K15 and K16, *AndBot* was used as $\curlywedge_2$, or the aggregation operator for results of implications in the implication term. However, this logical connective may not work as expected in practical. We explain this in the following paragraphs.

In an *n*-variable environment, *AndBot* in (20) can be generalized as follow:

$$\text{AndBot}(p_b) = \max(0, \sum_{b-1}^{n} p_b - (n-1)) \qquad (25)$$

for $b = 1, \cdots, n$. In K15 and K16, result of each implication is aggregated with (25), where p_b should be substituted with $R_{ab} \to S_{bc}$.

The problem associate with *AndBot* as the aggregation operator is not prominent when the number of *b* is small and results of $R_{ab} \to S_{bc}$ are big. However, it is easy to verify that when the number of *b* increases, the result of the aggregation decrease as long as the outcome of an implication is not 1.0. Some cases that are very likely to occur are as follow: 1) If all the 10 implications yield 0.9, the result of the aggregation is 0.0; 2) Out of *n* implications, if an implication yields 0.0, the result of the whole aggregation is also 0.0, even though all the other implications yield 1.0.

The tendency to reduce the result uni-directionally is not a desirable property of an aggregation operator in this sense. So, we conclude that K15 and K16, as well as any inference structures that use *AndBot* to aggregate the results of implications are not good inference structures practically.

Table 2. Values Generated by Implication Operators.

(a) $I_Ł(R,S)$

S / R	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.2	0.9	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.3	0.8	0.9	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.4	0.7	0.8	0.9	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.5	0.6	0.7	0.8	0.9	1.0	1.0	1.0	1.0	1.0	1.0
0.6	0.5	0.6	0.7	0.8	0.9	1.0	1.0	1.0	1.0	1.0
0.7	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.0	1.0	1.0
0.8	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.0	1.0
0.9	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.0
1.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

(b) $I_{KD}(R,S)$

S / R	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	1.0
0.2	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.9	1.0
0.3	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.8	0.9	1.0
0.4	0.6	0.6	0.6	0.6	0.6	0.6	0.7	0.8	0.9	1.0
0.5	0.5	0.5	0.5	0.5	0.5	0.6	0.7	0.8	0.9	1.0
0.6	0.4	0.4	0.4	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.7	0.3	0.3	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.8	0.2	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.9	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

4. The Influence of DeBaets and Kerre's Improvement on BK Sub-triangle Product

As discussed in Section 2, the validity of sub-triangle product is, in fact come from the subsethood measurement of one set in another, which provided by the implication term. Apparently, the additional term added by De Baets and Kerre to rectify the problem of emptiness has influenced in the subsethood measurement, despite it is the solution to non-emptiness condition.

Also, we learned in Section 3 that, taking *min* as $\curlywedge_1$ is to avoid the emptiness in R_{ab} to affect the result of inference. However, it is surprising that the additional term that proposed by De Baets and Kerre [13] generate smaller values in most cases, not only when the emptiness happened. This means that with the *min* outer connective, the sauce is better than the fish and the real subsethood measurements are ignored most of the time.

To study this influence in more detail, we consider the simplest instance where only 1 pair of *(R,S)* relations are involved. We listed 2 possible candidates of implication operators, $I_Ł$ and I_{KD} in Table 2(a) and Table 2(b) respectively. *AndTop* and *AndBot*, logical connectives that correspond to the additional term proposed by De Baets and Kerre in this case were listed in Table 3(a) and Table 3(b). Table 4 shows the difference between these 2 terms when the values of implications were override by additional term - those cells without any values represent the combination of *R* and *S* which the values of implications are smaller or equal to the additional terms.

From the tables, it is clear that the solution proposed by De Bates and Kerre has major influence on the inference structures in the case that $\curlywedge_1$ = *min* and there is only 1 pair of *(R,S)* relations. This cause the dilemma in choosing the outer connective since *max* is not a good connective either.

One may claim that the influence of additional term may decrease when the number of *(R,S)* relations increase. This is true because when the number of *(R,S)* relations increase, assigning $\curlyvee_3$ as Or operators, especially *OrBot* will result a relatively large value in the additional term, whereas And operators as $\curlywedge_1$ may cause the implication term to produce a smaller value. However, this is not enough to judge that the additional term will have no influence on the results of inferences.

Table 3. Values Generated by AndTop and AndBot.

(a) *AndTop*(R,S)

S / R	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
0.2	0.1	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
0.3	0.1	0.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
0.4	0.1	0.2	0.3	0.4	0.4	0.4	0.4	0.4	0.4	0.4
0.5	0.1	0.2	0.3	0.4	0.5	0.5	0.5	0.5	0.5	0.5
0.6	0.1	0.2	0.3	0.4	0.5	0.6	0.6	0.6	0.6	0.6
0.7	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.7	0.7	0.7
0.8	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.8	0.8
0.9	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.9
1.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

(b) *AndBot*(R,S)

S / R	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1
0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.2
0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.2	0.3
0.4	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.2	0.3	0.4
0.5	0.0	0.0	0.0	0.0	0.0	0.1	0.2	0.3	0.4	0.5
0.6	0.0	0.0	0.0	0.0	0.1	0.2	0.3	0.4	0.5	0.6
0.7	0.0	0.0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
0.8	0.0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.9	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

Table 4. Values Generated by AndTop and AndBot.

(a) *AndTop*(R,S)

R \ S	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
0.2	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
0.3	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
0.4	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.6	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
0.7	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
0.8	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
0.9	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
1.0										

(b) *AndBot*(R,S)

R \ S	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9
0.2	0.9	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9	0.8
0.3	0.8	0.9	1.0	1.0	1.0	1.0	1.0	0.9	0.8	0.7
0.4	0.7	0.8	0.9	1.0	1.0	1.0	0.9	0.8	0.7	0.6
0.5	0.6	0.7	0.8	0.9	1.0	1.0	0.8	0.7	0.6	0.5
0.6	0.5	0.6	0.7	0.8	0.1	0.8	0.7	0.6	0.5	0.4
0.7	0.4	0.5	0.6	0.6	0.2	0.6	0.6	0.5	0.4	0.3
0.8	0.3	0.4	0.4	0.4	0.3	0.4	0.4	0.4	0.3	0.2
0.9	0.2	0.2	0.2	0.2	0.4	0.2	0.2	0.2	0.2	0.1
1.0										

(c) *AndTop*(R,S)

R \ S	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.9
0.2	0.7	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.7	0.8
0.3	0.6	0.5	0.4	0.4	0.4	0.4	0.4	0.5	0.6	0.7
0.4	0.5	0.4	0.3	0.2	0.2	0.2	0.3	0.4	0.5	0.6
0.5	0.4	0.3	0.2	0.1		0.1	0.2	0.3	0.4	0.5
0.6	0.3	0.2	0.1				0.1	0.2	0.3	0.4
0.7	0.2	0.1						0.1	0.2	0.3
0.8	0.1								0.1	0.2
0.9										0.1
1.0										

(d) *AndBot*(R,S)

R \ S	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
0.2	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8
0.3	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
0.4	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
0.6	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
0.7	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
0.8	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
0.9	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
1.0										

5. Our Proposed Method

With the argument that the influence of additional term will be minor and tolerable once the number of *(R,S)* relations increases, it is fair to solve the problems by reconstruct the set of influence structures with reasonable logical connectives.

In this paper, we proposed this solution. Firstly, there should have no reason that *AndBot* must be kept as a candidate of $\curlywedge_2$. Subsequently, *max* should also be removed because of the reason explained in Section 3. Thus, the list of logical connective we have now is as follow:

$$\begin{aligned} \curlywedge_1 &= \{\text{min}\} \\ \curlywedge_2 &= \{\text{Arithemetic mean, AndTop}\} \\ \curlyvee_3 &= \{\text{Arithmetic mean, OrTop, OrBot}\} \\ \curlywedge_4 &= \{\text{AndTop, AndBot}\} \end{aligned} \tag{26}$$

With these logical connectives, we can generate a list of 12 sub-K inference structures. Moreover, as the best effort to minimize the influence of additional term, the set of logical connectives can be further reduced, especially $\curlyvee_3$, which is the main determiner of the additional term. To reduce the influence of additional term, the largest possible values should be generated by $\curlyvee_3$. From (21-22), we know that these logical connectives can be ranked as follow:

$$\text{Arithmetic mean} \le \text{OrTop}(p,q) \le \text{OrBot}(p,q)$$

So, the list of logical connectives further reduced to:

$$\begin{aligned} \curlywedge_1 &= \{\text{min}\} \\ \curlywedge_2 &= \{\text{Arithemetic mean, AndTop}\} \\ \curlyvee_3 &= \{\text{OrBot}\} \\ \curlywedge_4 &= \{\text{AndTop, AndBot}\} \end{aligned} \tag{27}$$

These logical connectives generate a list of 4 sub-K inference structures, namely K7, K9, K18 and K19. One can compare to Table 5 and find out that these are the top ranked (high MTA rate) inference structures in performance, along with BK2 and BK3. On the other hand, Sub-K inference structures that using *max* as outer connective, such as K3, K4, K5, K6, K8, K11 and K17 are among the highest MFA rate. All these inference structures have MFA from 0.56 to 0.96, due to the influence of additional term and ignorance of non-emptiness condition. The consistency to the experiment results shows that the theoretical discussions in previous sections are support by empirical work.

A. *Comparisons and Discussions*

It is worth to study the behavior of these inference structures in detail since they are among very few that without the shortcomings pointed out in Section 3. However, we don't have the original data used in [3] for any further analysis, so a set of simulation data as shown in Appendix I is prepared for this purpose.

These data was tested with K7, K9, K18 and K19. The same evaluation matrix is used. The results of the evaluation are presented in Table 6.

Table 5. Result of Inference using Test Data.

Inference Structure	MTA	MTR	MFA	MFR
K7	0.57	0.04	0.20	0.00
K9	0.57	0.01	0.20	0.00
K18	0.22	0.66	0.06	0.00
K19	0.22	0.66	0.06	0.00

Table 6. Comparing Performance of Inference Structures on Each Patient Group.

(a) Performance of K7

Patient Group	MTA	MTR	MFA	MFR
Group 1 (D_1)	1.00	0.00	0.00	0.00
Group 2 (D_2)	1.00	0.00	0.00	0.00
Group 3 (D_3)	0.33	0.1	0.50	0.00
Group 4 (D_4)	0.00	0.00	0.25	0.00
Group 5 (D_5)	0.50	0.05	0.25	0.00

(b) Performance of K9

Patient Group	MTA	MTR	MFA	MFR
Group 1 (D_1)	1.00	0.00	0.00	0.00
Group 2 (D_2)	1.00	0.00	0.00	0.00
Group 3 (D_3)	0.33	0.05	0.50	0.00
Group 4 (D_4)	0.00	0.00	0.25	0.00
Group 5 (D_5)	0.50	0.05	0.25	0.00

(c) Performance of K18

Patient Group	MTA	MTR	MFA	MFR
Group 1 (D_1)	0.00	0.60	0.00	0.00
Group 2 (D_2)	1.00	1.00	0.00	0.00
Group 3 (D_3)	0.10	0.50	0.05	0.00
Group 4 (D_4)	0.00	0.45	0.00	0.00
Group 5 (D_5)	0.00	0.75	0.25	0.00

(d) Performance of K19

Patient Group	MTA	MTR	MFA	MFR
Group 1 (D_1)	0.00	0.60	0.00	0.00
Group 2 (D_2)	1.00	1.00	0.00	0.00
Group 3 (D_3)	0.10	0.50	0.05	0.00
Group 4 (D_4)	0.00	0.45	0.00	0.00
Group 5 (D_5)	0.00	0.75	0.25	0.00

One must understand that it is meaningless to compare the score of inference structures in Table 5 with Table 1 directly, this is because different data is used. However, we can see the same pattern in both tables: K7 and K9 have high MTA, followed by K18 and then K19; all the MFR are 0; K19 and K18 has lower MFA compared to K7 and K9; K19 has higher MTR compared to K7 and K9.

When we go into the detail of the performance of each inference structures on each patient group (correspond ing to a disease), we found more useful information on the characteristic of these inference structures. We present these findings in Table 6.

Firstly, from Table 6(a-b) we found that K7 and K9 have quite similar performance. This is not surprising because the difference of K7 and K9 is only in their λ_4, which might be the least significant among all. The low influence of λ_4 is more prominent in the case of K18 and K18 (please refer to Table 6(c-d)), which both have the same score for all measurements.

Both K7 and K9 able to identify patients from Group 1 and Group 2 accurately. The correct diseases were identified and no incorrect diseases are accepted. However, for the acceptant rate of patients from Group 5, the performance drops for both inference structures. This is because D_2, the disease that resemble D_5 is also accepted in the inferencing. The same thing happened for D_3. Besides D_3, another 2 diseases, D_1 and D_4 which resemble D_3 are also accepted in these inferencing. For the inference of patients in Group 4, these inference structures just cannot identify the correct disease, but D_1 is given as the inference result.

For the rejection rate, we find that both K7 and K9 do not reject many diseases. Only P_{12-14} are identified as not victims of disease D_5. In other words, many diseases are left in the gray area.

So, it is clear that K7 and K9 do not performing good in rejecting diagnoses, but work well in identifying diseases when the diseases has clear characteristics (e.g. D_1 and D_2). For the identification of D_3, D_4 and D_5, the performance is not so good. The reason of this can be explained by studying the characteristic of sub-triangle product and the signs/symptoms of theses diseases. The sub-triangle product find the relation from patients to diseases by examining the subsethood of the afterset of patients in the foreset of diseases (refer to Definition 6). However, in these cases, foresets of D_3 and D_4 seems like a subset of foreset of D_1. So, if a patient's afterset is a subset of D_3 and D_4 afterset, it is also a subset of D_1. This characteristic of sub-triangle product caused the indistinguishable for diseases D_3 and D_4 from D_1. The problem faces by identification of D_5 can be explained in the same way.

Both K18 and K19 did not performed well in identifying correct diseases compared to K7 and K9. Yet, they shown great ability in rejecting false diagnoses. They rejected all the false diagnoses of patients from Group 2. Similar to true acceptance rate for K7 and K9, disease D_4 is the biggest problem to these inference structures. They are not able to reject D_1 and D_3 as correct diagnoses. However, they able to identify D_2 and D_5 as wrong diagnoses in most cases, except for patient P_{17}. Rejecting patients P_{1-5} and P_{11-15} from wrong diagnoses have similar problem because of high resemblance of disease D_3 and D_4 to D_1. On the other hand, these inference structures did not falsely rejected any diseases.

In previous work [3, 10], the test results are presented without further analysis on the performance and behaviour of the inference structures. In this paper, a detail analysis is performed and a better understanding on these inference structures is acquired. We can conclude that K7 and K9 have well performance in accepting correct diagnoses, whereas K18 and K19 performed good in rejecting wrong diagnoses. To form the inference engine of a medical expert system, a combination of K7/K9 and K18/K19 can be considered, which K7/K9 responsible to identify highly suspected diseases and K18/K19 responsible to filter out impossible diseases.

6. Conclusions

In this paper, we revisit a typical application that employed inference structures that developed from BK sub-triangle product. We found 2 shortcomings in the applications that related to the initialization of logical connectives. We also studied the influence of additional term that proposed by De Baets and Kerre to the BK sub-triangle product. Based on these findings, conclude that out of 19 inference structures developed from sub-K relational product used in [3, 10], only 4 are working well. These inference structures are K7, K9, K18 and K19. The conclusion of this theoretical analysis is also supported by the result of empirical work.

To have a better understanding on the performance of these inference structures. A set of simulation data is prepared, tested and evaluated with the same evaluation matrix as in [3, 10]. The result of this test is analogy to the result in [3, 10]. However, with more details on the test data that we have, we conclude that K7/K9 performed well in accepting correct diagnoses, whereas K18/K19 can reject impossible diagnoses even better. So, a combination of these inference structures should work well in forming inference engine of a medical expert system.

The imperfection of these inference structures was also found in the test with simulation data. When a disease with all its foreset of signs/symptoms resemblance the subset of another disease, inference structures derived from sub-K relational product unable to identify these diseases correctly. To solve this, we suggest that square product that seeking similarities between foresets of diseases and aftersets of patients should be studied. Of course, it may be too restrictive to perform medical inferences with similarity based inference structures. To cope with this problem, an upgrade to type-2 fuzzy theory with higher ability to deal with uncertainty is suggested.

Appendix I

A set of 5 diseases $\{D_1,\cdots D_5\}$ are designed in the list, each with 8 signs/symptoms $\{F_1,\cdots F_8\}$ (Table 7). To increase the difficulty of reasoning, the data, or relation between diseases and signs/symptoms are purposely designed like this: 1) D_1 and D_2 are 2 basic cases where D_1 shows high relations (0.7-1.0) to signs/symptoms F_{1-6}, whereas low relations (0.0 - 0.3) to F_{7-8}. Relatively, the D_2 shows low relations with first 2 signs/symptoms and high in the others. 2) D_{3-5} are diseases to test the ability of identification of inference structures. D_4 is closely resemble to D_1 except its 2 signs/symptoms, F_5 and F_6 showing moderate relations (0.4-0.6) instead of high. On the other hand, D_3 is also resemble D_4, as well as D_1. It shows low, instead of moderate or high in F_5 and F_6. Lastly, D_5 \$D_5\$ is resembling D_2 except its F_3 and F_5, which is low instead of high. Among all, F_5 and D_4 have highest similarity.

Table 7. Relations S –
Relation between Signs/Symptoms and Diseases.

Diseases / Signs	D_1	D_2	D_3	D_4	D_5
F_1	0.8	0.2	0.9	0.8	0.2
F_2	0.9	0.3	1.0	0.9	0.1
F_3	0.7	0.8	0.8	1.0	0.0
F_4	0.9	0.9	0.8	0.8	0.0
F_5	0.8	1.0	0.2	0.5	0.7
F_6	1.0	1.0	0.1	0.4	0.6
F_7	0.2	0.9	0.3	0.1	0.8
F_8	0.1	0.8	0.0	0.7	1.0

Table 8. Relations R –
Relation between Patient and Signs/Symptoms.

Signs / Patients	F_1	F_2	F_3	F_4	F_5	F_5	F_7	F_8
P_1	0.8	0.8	0.8	0.9	0.8	1.0	0.1	0.1
P_2	1.0	1.0	0.8	0.9	0.9	0.8	0.1	0.1
P_3	0.9	0.8	0.9	0.8	1.0	0.9	0.1	0.0
P_4	0.8	0.9	0.8	1.0	0.8	1.0	0.0	0.2
P_5	0.6	0.9	0.5	0.8	0.7	0.8	0.3	0.0
P_6	0.2	0.1	0.9	0.8	0.8	0.8	1.0	0.8
P_7	0.0	0.2	0.9	0.8	0.8	0.7	0.9	0.8
P_8	0.1	0.1	0.9	0.9	0.8	1.0	1.0	0.8
P_9	0.1	0.0	0.7	0.9	0.9	0.9	0.9	0.9
P_{10}	0.2	0.2	0.8	1.0	0.8	0.9	0.9	0.9
P_{11}	0.9	0.8	1.0	0.8	0.3	0.0	0.1	0.3
P_{12}	0.9	0.9	0.9	0.7	0.0	0.3	0.1	0.1
P_{13}	1.0	0.8	0.9	0.7	0.3	0.1	0.0	0.1
P_{14}	0.9	0.9	0.8	0.9	0.0	0.0	0.0	0.0
P_{15}	0.9	0.9	0.9	0.9	0.3	0.1	0.1	0.3
P_{16}	0.8	1.0	0.7	0.9	0.5	0.6	0.1	0.1
P_{17}	0.8	0.8	1.0	0.9	0.4	0.5	0.2	0.3
P_{18}	0.9	0.9	0.9	0.8	06	0.4	0.3	0.3
P_{19}	1.0	1.0	0.9	1.0	0.5	0.5	0.2	0.2
P_{20}	0.9	0.9	0.9	0.9	0.5	0.4	0.3	0.3
P_{21}	0.2	0.2	0.1	0.1	0.9	0.9	0.9	0.7
P_{22}	0.2	0.1	0.0	0.1	0.9	0.8	1.0	0.8
P_{23}	0.0	0.1	0.2	0.2	0.9	0.9	0.9	0.7
P_{24}	0.2	0.2	0.0	0.1	1.0	0.8	0.9	0.7
P_{25}	0.1	0.0	0.2	0.0	0.9	0.7	0.8	0.9

Acknowledgment

This research is supported by the University of Malaya HIR under grant No UM.C/625/1/HIR/037, J0000073579.

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