An Inference Engine Based on Interval Type-2 Fuzzy BK Subproduct

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Abstract—Recently, Bandler-Kohout (BK) subproduct based reasoning scheme has been a popular choice in various kind of applications. In this paper, we aim to enhance the BK subproduct based reasoning schemes in two aspects: (1) Extend the BK subproduct in term of Interval Type-2 Fuzzy Sets (IT2FS) instead of the ordinary Type-1 Fuzzy Sets (T1FS), and (2) Introduce weight parameter to the reasoning scheme. Firstly, studies have shown that IT2FS have better capability in handling data with uncertainty compare to the ordinary T1FS. Thus, we extend the BK subproduct in terms of IT2FS theory where subsethood measure based on the fuzzy implication operators for the IT2FS has been developed. Secondly, weight parameter associated to each features is introduced to form a weighted inference scheme with the BK subproduct. The introduction of the weight parameter is to aid in distinguishing the influence of different features in the reasoning process. In here, the Linguistic Weighted Average (LWA) is adopted to solve the outputs of this weighted reasoning scheme. Finally, a case study is employed to demonstrate the capability of the proposed approach.

I. INTRODUCTION

Recent studies [1], [2] have risen the excellency of fuzzy implication based deductive reasoning schemes, such as the Bandler-Kohout (BK) subproduct [3], [4]. These studies showed the BK subproduct based reasoning scheme able to demonstrate and match all the important properties of the much more popular reasoning scheme, namely the Compositional Rule of Inference (CRI) [5]. This has deducted that BK subproduct is competent to form effective and efficient inference engines if appropriate rules are given.

Apart from the rule based reasoning discussed in [1], [2], BK subproduct also well known with its great achievements in case based reasoning. Some successful examples that can be found in the literature include medical expert systems [6], [7], intelligent navigation systems [8], information retrieval [9], land evaluation [10], cognitive sciences [11], scene understanding [12] and etc.

Despite of its successfulness, there are some limitations associated in the current BK subproduct in case based reasoning. First of all, the implementations of BK subproduct in the literature are still based on classical Type-1 Fuzzy Sets (T1FS) theory, which address uncertainty with point-values. Studies such as [13], [14] claimed that T1FS has its limitation in addressing uncertainty with its crisp membership functions. To improve the uncertainty handling of BK subproduct-based inference engines, an extension to Interval Type-2 Fuzzy

Sets (IT2FS) [15], [16] is needed. Thus, in this paper, our first contribution is to propose and extend the current BK subproduct to IT2FS.

Secondly, BK subproduct performs inferences by utilizing a set of common features that relate the inputs and outputs. In most cases, the BK subproduct treats all the features equally, i.e. the importance of all features are similar. However, in practical, not all these features have the same influences towards inference results, we argue that some features may have higher reliability or distinguishability than the others, and vice versa. Thus, our second contribution is to introduce weight parameter in the BK subproduct.

Work that similar to us in this case is [10], however, the implementation of the weight in this work required to fullfill a condition: $\sum_{i=1}^{N} w_i = 1$ where N is the number of features and w is the weight of feature i. This condition is too restrictive for good implementation of weight because: *(i)* adding or decreasing features into consideration list will cause recalculation of all the weights. For instance, adding a new feature with weight $w_{N+1} \neq 0$ to existing feature list will cause the total weight become $\sum_{i=1}^{N+1} w_i$. It is easy to verify that $\sum_{i=1}^{N+1} w_i = 1 + w_{N+1} > 1$ and the condition of total weight equal to 1 is not longer valid. Thus, a normalization is required so that the $\sum_{i=1}^{N+1} w_i = 1$ is fulfilled. *(ii)* importance or influence of a feature is not intuitive - i.e. comparing a system with such condition to an implementation of weight where $w_i \in [0, 1]$ for all i, the weights of the latter are much intuitive as weights close to 0 means less influence, while close to 1 means high influence. In [10], the weights can be small numbers that close to 0 even if they have high influence in the case of N is a big number. Furthermore, this problem become much more complicated if new features are going to be added into consideration as one may not know what are the appropriate values that representing high (or low) influence. In contrast to [10], our weight term is $w_i \in [0, 1]$ and condition: $\sum_{i=1}^{N} w_i = 1$ is not required.

The rest of the paper are arranged as follow: In Section II, we provide a short revision on BK subproduct. Section III discusses the extension of BK subproduct from T1FS to IT2FS, along with a subsethood measure of IT2FS. In Section IV, we introduce the weight parameter to this newly developed IT2FS reasoning scheme. A case study is presented in Section V for demonstration purpose. Lastly, we conclude the paper

in Section VI.

II. BACKGROUND

We start this section with a brief review on the definition of BK subproduct on crisp relations.

Assume that A, B and C are three crisp sets and a, b and c are general representation to the elements in the sets respectively. R is defined as a relation from A to B such that $R \subseteq A \times B$; whereas S is a relation from B to C such that $S \subseteq B \times C$. The converse relation of S is denoted as S^T. The abbreviation aRb shows that a is in relation R with b. Bandler and Kohout [4] defined (crisp) BK subproduct as follow:

Definition 1. BK subproduct is a composition of relations for a and c such that:

$$R \lhd S = \{(a,c)|(a,c) \in A \times C \text{ and } aR \subseteq Sc\} \quad (1)$$

where $aR = \{ b \mid aRb \}$ is the image of a after the projection of relation R in the set B, while $Sc = \{ b \mid bSc \}$ is the image of c after the projection of relation S^T in the set B. BK subproduct is useful in retrieving relationships between elements of two indirectly associated sets, *objects* A and *targets* C, if both sets can be associated with a set of common *features*, B.

Example 1 (Medical Inference Systems). Consider a set of Patients A, a set of Diseases C, and a set of Signs/Symptoms B. R is the relation from A to B indicating the signs/symptoms shown by patients, S is a relation from B to C, indicating the signs/symptoms used to characterize the diseases. Hence, BK subproduct $a'(R \lhd S)$ provides all diseases that may show all signs/symptoms of patient a'. In another word, all the signs/symptoms of patient a' are subset(s) of signs/symptoms of the disease(s) c.

It is trivial that **Definition 1** is established on the subsethood of aR in Sc. In order to extend it to fuzzy relations, the fuzzy subsethood measure for T1FS is defined as:

Definition 2. For two T1FS, P and Q in the same universe X, the possibility of P is a subset of Q is given as follow:

$$\pi(P \subseteq Q) = \bigwedge_{x \in X} (\mu_P(x) \to \mu_Q(x)) \quad (2)$$

where $\to$ is fuzzy implication operators generally defined as "NOT A OR B" and $\wedge$ is the infimum operator. $\wedge$ can be considered as $\min$ *function* in harsh criterion or *arithmetic mean* in mean criterion [4].

Definition 3. With Eq. (1) and Eq. (2), Bandler and Kohout [4] defined the fuzzy BK subproduct as follow:

$$R \lhd S(a,c) = \bigwedge_{b \in B} (R_{ab} \to S_{bc}) \quad (3)$$

where R_{ab} is the fuzzy degree to which aRb is true and S_{bc} is the fuzzy degree to which bSc is true.

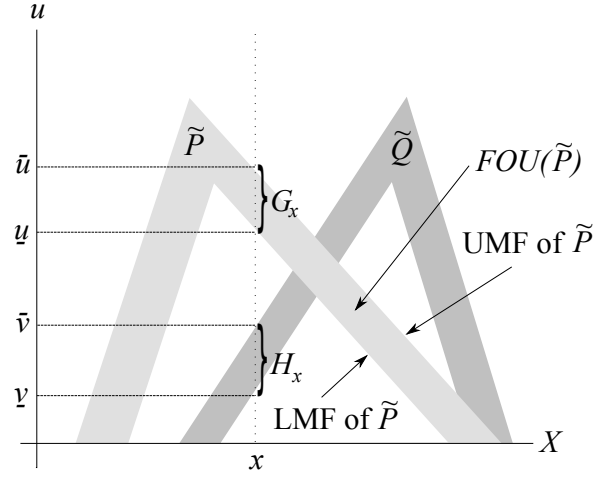Later, De Baets and Kerre [17] discovered that the BK subproduct demonstrated a limitation in a specific condition and

Fig. 1.   Two IT2FS $\tilde{P}$ and $\tilde{Q}$ in the same universe X. The primary memberships for $\tilde{P}$ and $\tilde{Q}$ are denoted as G_x and H_x respectively. For G_x, u denotes its secondary variable whereas $\underline{u}$ and $\overline{u}$ are its lower and upper limits respectively. The area in light gray is the Footprint of Uncertainty of $\tilde{P}$ and its lower and upper membership functions (LMF and UMF) are labeled in the figure.

improvement has been proposed by incorporating additional term. However, recent work [18] discovered that the additional term gives unwanted influence to the inference results. Thus, in this paper, the original version of fuzzy BK subproduct is employed as in [10].

III. BK SUBPRODUCT WITH INTERVAL TYPE-2 FUZZY SETS

A. Interval Type-2 Fuzzy Sets

An Interval Type-2 Fuzzy Set (IT2FS) is a special case of a Type-2 Fuzzy Set which the membership functions of an element is not point-value but an interval [15], [16].

An IT2FS denoted as $\tilde{P}$ in universe X has elements $x \in \tilde{P}$. The membership functions of x are intervals G_x. We also name these intervals as primary memberships. The lower and upper endpoints of a G_x are given by an interval $[\underline{u}, \overline{u}]$ where u are called secondary variables.

If we depict all the primary memberships of $x \in \tilde{P}$ in a 2D graph, the area covered by those primary memberships is the Footprint of Uncertainty (FOU) of $\tilde{P}$, denoted as $FOU(\tilde{P})$. The outer margin of the FOU is the upper bound of FOU, called the Upper Membership Function (UMF), and the lower margin of the FOU is the lower bound of FOU, called the Lower Membership Function (LMF). An illustration is provided in Fig. 1.

B. Subsethood Measure of IT2FS

As fuzzy implication operators are only defined for point values, Eq. (2) needs to be extended so that the subsethood measurements for IT2FS can be computed. For this purpose, we incorporate the Representation Theorem [16].

Representation Theorem [16] suggests that an IT2FS can be represented as a collection of embedded T1FS. Assume that both axes x and u are discrete, an element $x_i \in \tilde{P}$ where

$i = \{1, 2, \cdots, I\}$ has J_i discrete secondary variables in its FOU, namely u_{ij} where $j = \{1, 2, \cdots, J_i\}$. The number of embedded T1FS in $\tilde{P}$ is:

$$\eta_{\tilde{P}} = \prod_{i=1}^{I} J_i \qquad (4)$$

Since these embedded T1FS are distributed evenly, the number of embedded T1FS pass through a secondary variable u_{ij} is $\frac{\eta_{\tilde{P}}}{J_i}$.

In order to define the fuzzy subsethood measure, or the possibility $\tilde{P}$ is a subset of $\tilde{Q}$, i.e. $\pi(\tilde{P} \subseteq \tilde{Q})$, we use the notations of set $\tilde{Q}$ as shown in Fig. 1: the primary membership of $x \in \tilde{Q}$ is H_x and secondary variable v, therefore $H_x = [\underline{v}, \overline{v}]$. For an element x_i, $i = \{1, 2, \cdots, I\}$, K_i is the number of discrete variables v_{ik} where $k = \{1, 2, \cdots, K_i\}$. The total number of embedded T1FS in $\tilde{Q}$ is given by $\eta_{\tilde{Q}} = \prod_{i=1}^{I} K_i$ and number of embedded T1FS on a secondary variable v_{ik} is $\frac{\eta_{\tilde{Q}}}{K_i}$.

To formulate the fuzzy subsethood measure of IT2FS, we start with evaluating an arbitrary pair of secondary variables u_{ij} and v_{ik} in $\tilde{P}$ and $\tilde{Q}$ respectively, on a same element x_i. The implication of these points to the subsethood measure $\pi(\tilde{P} \subseteq \tilde{Q})$ is $u_{ij} \rightarrow v_{ik}$. However, since there are $\frac{\eta_{\tilde{P}}}{J_i}$ of embedded T1FS on u_{ij} and $\frac{\eta_{\tilde{Q}}}{K_i}$ on v_{ik}, this implication involves a number of $\frac{\eta_{\tilde{P}}}{J_i} \times \frac{\eta_{\tilde{Q}}}{K_i}$ pairs of embedded T1FS, thus it should be represented as:

$$\frac{\eta_{\tilde{P}} \eta_{\tilde{Q}}}{J_i K_i} \left(u_{ij} \rightarrow v_{ik} \right) \qquad (5)$$

If u_{ij} is the only discrete point in G_x, the subsethood measure can be evaluate by summing up the implication of this secondary variable to all the v_{ij} (Fig 2):

$$\frac{\eta_{\tilde{P}} \eta_{\tilde{Q}}}{J_i K_i} \sum_{k=1}^{K_i} \left(u_{ij} \rightarrow v_{ik} \right) \qquad (6)$$

In general cases, G_x are intervals with more then one discrete points. Therefore, if we generalized Eq. (6) to all the u_{ij} in the element and normalized it with the total number of embedded T1FS $\eta_{\tilde{P}} \eta_{\tilde{Q}}$, we have the fuzzy subsethood measure for the element x_i:

$$\pi(\tilde{P} \subseteq \tilde{Q})(x_i) = \frac{1}{J_i K_i} \sum_{j=1}^{J_i} \sum_{k=1}^{K_i} \left(u_{ij} \rightarrow v_{ik} \right) \qquad (7)$$

If we substitute Eq. (7) to Eq. (2) and the mean criterion is considered, the complete fuzzy subsethood measure for IT2FS is:

$$\pi(\tilde{P} \subseteq \tilde{Q}) = \frac{1}{I} \sum_{i=1}^{I} \frac{1}{J_i K_i} \sum_{j=1}^{J_i} \sum_{k=1}^{K_i} \left(u_{ij} \rightarrow v_{ik} \right) \qquad (8)$$

Employing a fuzzy implication operator, Eq. (8) will produce a subsethood measurement of $\tilde{P} \subseteq \tilde{Q}$ in interval $[0, 1]$. However, we noticed that the subsethood measurements for crisp sets are Boolean (yes / no) and for T1FS are point-values. Therefore, we deduce that the fuzzy subsethood measurements for IT2FS should be intervals instead of point-values as of
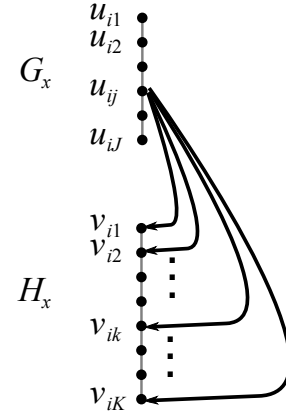

Fig. 2. Implication of a single discrete secondary variable in primary membership G_x of $\tilde{P}$ to all the discretized secondary variables in primary membership H_x of $\tilde{Q}$

[19]–[22]. Studies in [3], [23], [24] suggested that Kleene-Dienes and Łukasiewicz implication operators are among the two most suitable candidates for lower and upper bounds subsethood measurements. Thus, if $Y = [\underline{Y}, \overline{Y}]$ denotes lower and upper bounds of a subsethood measure:

$$\underline{Y} = \frac{1}{I} \sum_{i=1}^{I} \frac{1}{J_i K_i} \sum_{j=1}^{J_i} \sum_{k=1}^{K_i} \left(u_{ij} \rightarrow_{\text{KD}} v_{ik} \right) \qquad (9)$$

$$\overline{Y} = \frac{1}{I} \sum_{i=1}^{I} \frac{1}{J_i K_i} \sum_{j=1}^{J_i} \sum_{k=1}^{K_i} \left(u_{ij} \rightarrow_{\text{Ł}} v_{ik} \right) \qquad (10)$$

where $\rightarrow_{\text{KD}}$ and $\rightarrow_{\text{Ł}}$ are Kleene-Dienes and Łukasiewicz implication operators respectively:

$$\alpha \rightarrow_{\text{KD}} \gamma = \max \left(\gamma, 1 - \alpha \right) \qquad (11)$$

$$\alpha \rightarrow_{\text{Ł}} \gamma = \min \left(1, 1 - \alpha + \gamma \right) \qquad (12)$$

C. Extension of Fuzzy BK Subproduct

Definition 1, 2 and 3 clearly showed that subsethood measure is the underlying idea of BK subproduct. Definition of BK subproduct is subsethood measure of aR (image of an object a in feature set B under relation R) in another set Sc (image of a target c in feature set B under converse relation S).

Thus, BK subproduct in term of IT2FS is ease to develop once we able to retrieve the subsethood measure of IT2FS. It is reasonable to assume that $\tilde{P}$ as $a\tilde{R}$ and $\tilde{Q}$ as $\tilde{S}c$. Due to the similarity between BK subproduct and subsethood measure, the notations used in the subsethood measure are used in BK subproduct as well. I is the number of elements of B. J_i is the interval that representing membership grade of $a\tilde{R}b_i$ and u_{ij} are discretized values in the interval. Similarly, K_i is the interval that representing membership grade of $b_i\tilde{S}c$ and v_{ik} are discretized values in K_i:

$$\tilde{R} \triangleleft \tilde{S}(a, c) = \frac{1}{I} \sum_{i=1}^{I} \frac{1}{J_i K_i} \sum_{j=1}^{J_i} \sum_{k=1}^{K_i} \left(R_{ab_{ij}} \rightarrow S_{b_{ik}c} \right) \qquad (13)$$

Similar to Eq. (8), the results of Eq. (13) are intervals too. If the lower and upper bounds of these intervals are denoted as $\tilde{R}_{\underline{\lhd}}\tilde{S}$ and $\tilde{R}_{\overline{\lhd}}\tilde{S}$ respectively, with Eq. (11) and Eq. (12), we obtain:

$$\tilde{R}_{\underline{\lhd}}\tilde{S}(a,c) = \frac{1}{I}\sum_{i=1}^{I}\frac{1}{J_iK_i}\sum_{j=1}^{J_i}\sum_{k=1}^{K_i}\max(S_{b_{ik}c}, 1-R_{ab_{ij}}) \quad (14)$$

$$\tilde{R}_{\overline{\lhd}}\tilde{S}(a,c) = \frac{1}{I}\sum_{i=1}^{I}\frac{1}{J_iK_i}\sum_{j=1}^{J_i}\sum_{k=1}^{K_i}\min(1, 1-R_{ab_{ij}}+S_{b_{ik}c}) \quad (15)$$

IV. WEIGHT IN BK SUBPRODUCT BASED INFERENCE ENGINE

A. Weight and BK Subproduct

In most of the time, we can group criteria that we need to consider into a few criteria sets during reasoning. Among these criteria sets, some of them might have higher influence over the others in a decision making process. For instance, let us look back to Example 1. A physician may consider the following 4 criteria sets during a medical diagnosis, namely symptoms, patient personal history, family history and environmental issues. However, not all the criteria sets having the same influence in the medical diagnosis. In the diagnosis of diseases such as breast cancer, physicians may take more consideration on symptoms (higher influence) found on patients compare to environmental issues (low influence). Hence, weights are meaningful in representing the influence of the criteria sets.

However, one should note that the weight should not be confused with the strength of criteria in the criteria set. Similar case is applied to inference engines based on BK subproduct. Instead of criteria, features are considered here. Membership functions of objects-features relations ($\tilde{R}$) and features-target relations ($\tilde{S}$) are the "strength of criteria" that determine the results of inferencing. While features forming feature sets, influence of each feature sets are represented as weight. Weight applies to each feature set and we model the weight with IT2FS in this paper.

As mentioned earlier, subsethood measure is the fundamental of fuzzy BK subproduct. Thus, one might argue that it is inappropriate to implement weight in BK subproduct based inference engines because there is no well defined weighted subsethood measure in the literature. In fact, the weights are applied to the feature sets rather then the subsethood measurements. We explain the argument with a multiple feature sets model explained below.

Assume that the features in set B can be grouped into multiple feature sets B_m, $m = \{1, 2, \cdots, M\}$. Each feature set has a number of features. The relation between A and B_m is $\tilde{R}_m$, whereas $\tilde{S}_m$ is the relation between B_m and C. In this case, the images of $a\tilde{R}_m$ and $\tilde{S}_mc$ are $\tilde{P}_m$ and $\tilde{Q}_m$ respectively. Subsethood measure of $\tilde{P}_m \subseteq \tilde{Q}_m$ gives $\tilde{R}_m \lhd \tilde{S}_m(a,c)$ (Fig. 3).
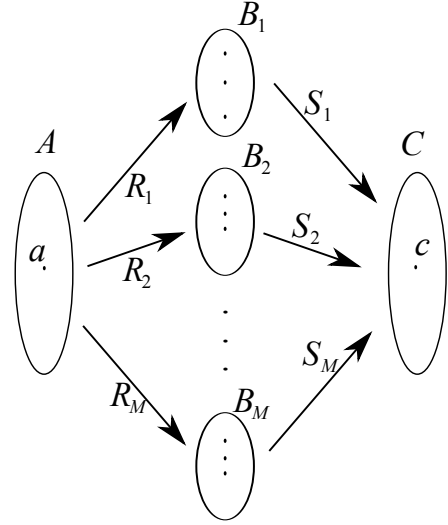

Fig. 3. Breaking up set B to multiple feature sets to form weighted BK subproduct.

Each feature set carries different weight. Assume that the weight of $\tilde{R}_m \lhd \tilde{S}_m(a,c)$ is $\tilde{W}_m$, the normalized aggregation of all the composition of relations is given as:

$$\tilde{R} \lhd \tilde{S}(a,c) = \frac{\displaystyle\sum_{m=1}^{M}\tilde{W}_m\left(\tilde{R}_m \lhd \tilde{S}_m(a,c)\right)}{\displaystyle\sum_{m=1}^{M}\tilde{W}_m} \quad (16)$$

Eq. (16) gives the weighted measure of BK subproduct. Here, since all $\tilde{R}_m \lhd \tilde{S}_m(a,c)$ are intervals that only exist as numerators, whereas $\tilde{W}_m$ are IT2FS, the results of computations based on Eq. (16) are always IT2FS. We show the details of solving Eq. (16) in the following subsection.

B. Solving the Weighted Average

Solving Eq. (16) is easy if all the parameters are crisp numbers. However, these parameters are fuzzy, and so the solution become slightly complicated, especially with a term $1/\sum_{m=1}^{M}\tilde{W}_m$. One of the closest problem as us that solved in the literature is Fuzzy Weighted Average (FWA) [25]. FWA solved the problems in the form of:

$$f = \sum_{m=1}^{M}\omega_m\chi_m \bigg/ \sum_{m=1}^{M}\omega_m \quad (17)$$

where all χ_m and ω_m are T1FS. Wu and Mendel [26], [27] extended FWA to form Linguistic Weighted Average (LWA), which solve the problem where all χ_m and ω_m are IT2FS. Both FWA and LWA uses α-cut decomposition theorem [28] in solving the problems. With α-cut decomposition theorem, instead of performing calculation directly on the sets (χ_m and ω_m) as whole, a number of $(\delta-1)$ α-cuts are taken to break the sets into δ intervals. For each interval α_ι, $1 \leq \iota \leq \delta$, perform calculation that suppose to for the sets to yield an

R_{ab} or S_{bc}	a_1	a_2	a_3	c_1
b_{11}	[0.1 , 0.3]	[0.9 , 1.0]	[0.4 , 0.5]	[0.2 , 0.3]
b_{12}	[0.2 , 0.3]	[0.8 , 0.9]	[0.7 , 0.8]	[0.1 , 0.2]
b_{13}	[0.3 , 0.5]	[0.8 , 1.0]	[0.6 , 0.9]	[0.4 , 0.4]
b_{14}	[0.7 , 0.8]	[0.8 , 1.0]	[0.8 , 0.9]	[0.7 , 0.8]
b_{21}	[0.9 , 1.0]	[0.2 , 0.2]	[0.7 , 0.8]	[0.2 , 0.3]
b_{22}	[0.6 , 0.7]	[0.6 , 0.7]	[0.7 , 0.8]	[0.4 , 0.6]
b_{31}	[0.7 , 0.9]	[0.1 , 0.3]	[0.7 , 0.8]	[0.1 , 0.2]
b_{32}	[0.8 , 0.9]	[0.1 , 0.2]	[0.7 , 0.8]	[0.1 , 0.2]
b_{33}	[0.6 , 0.7]	[0.2 , 0.5]	[0.1 , 0.1]	[0.3 , 0.4]
b_{41}	[0.9 , 1.0]	[0.2 , 0.2]	[0.0 , 0.1]	[0.1 , 0.2]
b_{42}	[0.8 , 0.9]	[0.5 , 0.8]	[0.1 , 0.1]	[0.7 , 0.8]

	a_1	a_2	a_3
c_1	[0.48, 0.68]	[0.61, 0.78]	[0.54, 0.71]

$$z_4 = \frac{\sum_{m=1}^{\beta_4} w_{m1}\overline{Z}_m + \sum_{m=\beta_4+1}^{M} w_{m4}\overline{Z}_m}{\sum_{m=1}^{\beta_4} w_{m1} + \sum_{m=\beta_4+1}^{M} w_{m4}} \quad (21)$$

In these equations, β_1, β_2, β_3 and β_4 are the switching points in the range $[1, M]$ calculated with Karnik-Mendel algorithm [29], [30] such that:

$$\underline{Z}_{\beta_1} \le z_1 \le \underline{Z}_{\beta_1+1} \quad (22)$$

$$\underline{Z}_{\beta_2} \le z_2 \le \underline{Z}_{\beta_2+1} \quad (23)$$

$$\overline{Z}_{\beta_3} \le z_3 \le \overline{Z}_{\beta_3+1} \quad (24)$$

$$\overline{Z}_{\beta_4} \le z_4 \le \overline{Z}_{\beta_4+1} \quad (25)$$

C. Results Interpretation

Compositions of the results obtained from Eq. (18)-(21) give a set of IT2FS, and the meaning carried by this set of IT2FS is application dependent. In some applications, ranking algorithms suggested in [31], [32] are useful if comparisons between these IT2FS are needed to find the highest order one. Engineering control applications might fall to this category. In some other cases, the results can be compared with a set of predefined IT2FS using similarity measure [19], [32].

V. CASE STUDY

In this section, we extend Example 1 to a case study. The purpose of this is to demonstrate the computation procedure and the influence of weights in inferencing.

Consider $A = \{a_1, a_2, a_3\}$ as a set of three patients and $C = \{c_1\}$ as a set with 1 disease. B is a set of 11 features that can be used in this medical diagnosis. This features are grouped into 4 sub feature sets (e.g: sign/symptoms, patient history, family history, environmental issues) such that : $B_1 = \{b_{11}, b_{12}, b_{13}, b_{14}\}$, $B_2 = \{b_{21}, b_{22}\}$, $B_3 = \{b_{31}, b_{32}, b_{33}\}$ and $B_4 = \{b_{41}, b_{42}\}$. Table I shows the relations between A, B and C. The weights of B_1, B_2, B_3 and B_4 are W_1, W_2, W_3 and W_4 respectively. Refer to Fig. 5 and Table IV for details of these weights.

A. Results

If weights are not considered in this case, grouping all features into multiple feature sets is meaningless. Thus, just consider that B is a set with all the 11 features, applying Eq. (14) and Eq. (15) give the results presented in Table II.

To consider the weights, firstly we compute the inference results for each group of features using Eq. (14) and Eq. (15). The results are presented in Table III.

As described in Section IV-B, solving this weighted BK subproduct based inferencing need multiple α-cuts. However, since all the membership functions in this case are trapezoidal, three α or strong α-cuts is effective to compute the answer, i.e. α=0+, 0.6 and 1.
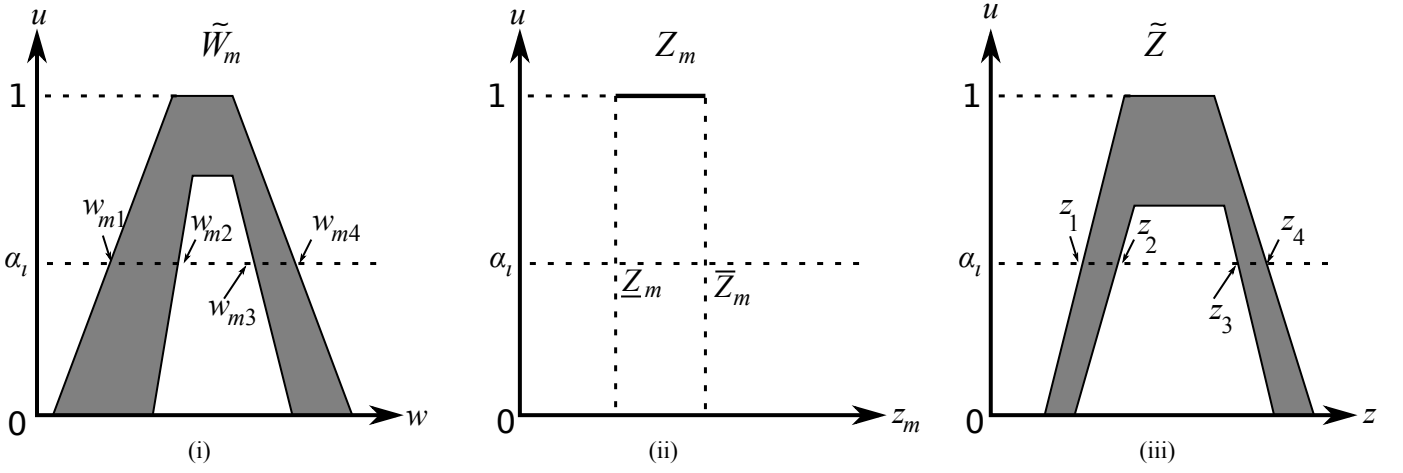
interval f_ι. The composition of all the f_ι with corresponding α-cuts will form the corresponding set f.

With the existing findings, Eq. (16) can be solved easily. We adopted the solution of LWA in [26], [27] by assuming that the intervals $\tilde{R}_m \triangleleft \tilde{S}_m(a,c)$ in Eq. (16) are special cases of IT2FS, where these fuzzy sets have rectangle membership functions and the UMF($\tilde{R}_m \triangleleft \tilde{S}_m(a,c)$) = LMF($\tilde{R}_m \triangleleft \tilde{S}_m(a,c)$). From here onwards, we denote $\tilde{R}_m \triangleleft \tilde{S}_m(a,c)$ as Z_m and the lower and upper bounds of Z_m are denoted as $\underline{Z}_m$ and $\overline{Z}_m$ respectively. Thus, follow this notation scheme, $\tilde{R} \triangleleft \tilde{S}(a,c)$ is denoted as $\tilde{Z}$.

Since the FOU of $\tilde{Z}$ is determined by UMF($\tilde{Z}$) and LMF($\tilde{Z}$), we can find $\tilde{Z}$ by calculating these two boundaries only. In [27], Wu and Mendel proved that the height of the output sets from LWA are equal to the minimum height of all Z_m and W_m. In our case, since all UMF($\tilde{W}_m$) are normal and Z_m are intervals, the height of an UMF($\tilde{Z}$) is unity. On the other hand, the height of a LMF($\tilde{Z}$) is totally depends on LMF($\tilde{W}_m$). Assume that all the $\tilde{W}_m$ are having trapezoidal (or triangular) shape FOU, the shape of $\tilde{Z}$ should be trapezoidal (or triangular) as well (Fig. 4).

As described earlier in this section, the solution of Eq. (16) starts with taking $(\delta - 1)$ α-cuts to yield δ intervals for each set. Thus, the rest of the work is simplified to find the intervals that represent the FOU of $\tilde{Z}$ corresponding to each α-cut. For this purpose, we refer to the notations that described in Fig. 4.

By referring to the results of LWA [26], [27], for each α-cut, the corresponding boundaries of UMF($\tilde{Z}$) and LMF($\tilde{Z}$) can be obtained by sorting $\overline{Z}_m$ and $\underline{Z}_m$ first, then substituting the corresponding values into the following equations:

$$z_1 = \frac{\sum_{m=1}^{\beta_1} w_{m4}\underline{Z}_m + \sum_{m=\beta_1+1}^{M} w_{m1}\underline{Z}_m}{\sum_{m=1}^{\beta_1} w_{m4} + \sum_{m=\beta_1+1}^{M} w_{m1}} \quad (18)$$

$$z_2 = \frac{\sum_{m=1}^{\beta_2} w_{m3}\underline{Z}_m + \sum_{m=\beta_2+1}^{M} w_{m2}\underline{Z}_m}{\sum_{m=1}^{\beta_2} w_{m3} + \sum_{m=\beta_2+1}^{M} w_{m2}} \quad (19)$$

$$z_3 = \frac{\sum_{m=1}^{\beta_3} w_{m2}\overline{Z}_m + \sum_{m=\beta_3+1}^{M} w_{m3}\overline{Z}_m}{\sum_{m=1}^{\beta_3} w_{m2} + \sum_{m=\beta_3+1}^{M} w_{m3}} \quad (20)$$

Fig. 4. Shape and notations of sets. (i) $\tilde{W}_m$: for an α-cut α_ι, $w_{\iota m1}$ and $w_{\iota m4}$ should be the leftmost and rightmost values of UMF($\tilde{W}_m$) respectively at α_ι. However, we leave out the variable ι as an subscript of all variables here and follows because it is independent from the calculation of each iteration, and to make the equations look more concise. Therefore, these variables become w_{m1} and w_{m4}. Similarly, w_{m2} and w_{m3} are the leftmost and rightmost values of LMF($\tilde{W}_m$) respectively. (ii) Z_m : $\underline{Z}_m$ is the lower bound of interval Z_m, whereas $\overline{Z}_m$ is the upper bound of this interval. (iii) $\tilde{Z}$: for an α-cut α_ι, z_1 and z_4 are the leftmost and rightmost values of UMF($\tilde{Z}$) respectively. Similarly, z_2 and z_3 are the leftmost and rightmost values of LMF($\tilde{Z}$) respectively.

TABLE III
INFERENCE RESULTS OF EACH GROUP OF FEATURES FOR EACH PATIENT.

	Z_1	Z_2	Z_3	Z_4
a_1	[0.73,0.96]	[0.38,0.58]	[0.26,0.45]	[0.45,0.55]
a_2	[0.39,0.49]	[0.65,0.93]	[0.77,0.95]	[0.78,0.97]
a_3	[0.49,0.69]	[0.39,0.63]	[0.47,0.60]	[0.93,1.00]

TABLE IV
THIS TABLE PRESENTS THE NUMERICAL VALUES OF THE PARAMETERS
THAT FORMED WEIGHTS IN FIG. 5.

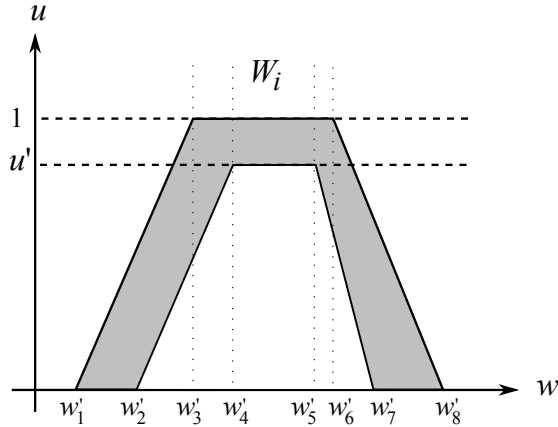|	u'	w'_1	w'_2	w'_3	w'_4	w'_5	w'_6	w'_7	w'_8
W_1	0.70	0.70	0.80	0.85	0.95	1.00	1.00	1.00	1.00
W_2	0.60	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
W_3	0.70	0.00	0.00	0.00	0.00	0.00	0.10	0.15	0.25
W_4	0.80	0.00	0.10	0.20	0.20	0.20	0.25	0.35	0.40

Fig. 5. A weight and its parameters modeled as IT2FS with trapezoidal membership function.

With Eq. (18) and Eq. (21), UMF($\tilde{Z}$) can be computed, whereas Eq. (19) and Eq. (20) give the results of LMF($\tilde{Z}$). These results are presented in Fig. 6. To make the comparison with Table II easier, interval results in Table II are presented as a "**black** bar" in Fig. 6.

B. Discussion

In Fig. 6(a), although defuzzification is needed so that the weighted inference result (represented with IT2FS) can be compared with its unweighted result (represented with interval), but it is trivial that the weighted inference result gives stronger suggestion that the patient a_1 is suffering with disease c_1.
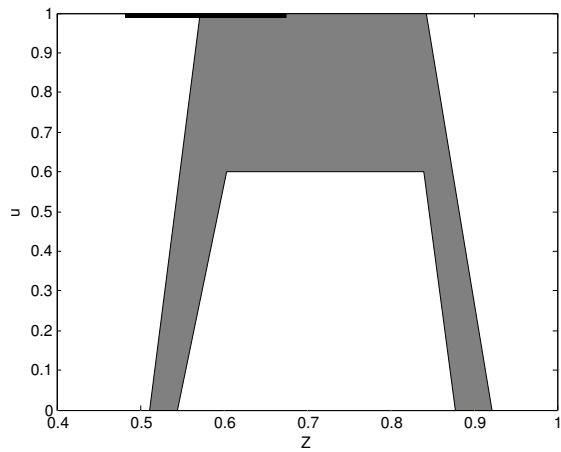
On the other hand, Fig. 6(b) shows a different scenario. Compare to the result of weighted inference, the unweighted result gives stronger suggestion that patient a_2 is suffering with the illness. The cases of these two patients, a_1 and a_2, clearly showed that the influence of weight parameter in inferencing.

For the case of patient a_3, we can observe the following: (i) with the influence of weight, both the results have higher coherency compare to the cases of a_1 and a_2; (ii) although its unweighted result is quite similar to the unweighted result of of a_1, but their weighted results give different impression. Comparatively, its weighted result is more similar to the weighted result of patient a_2.
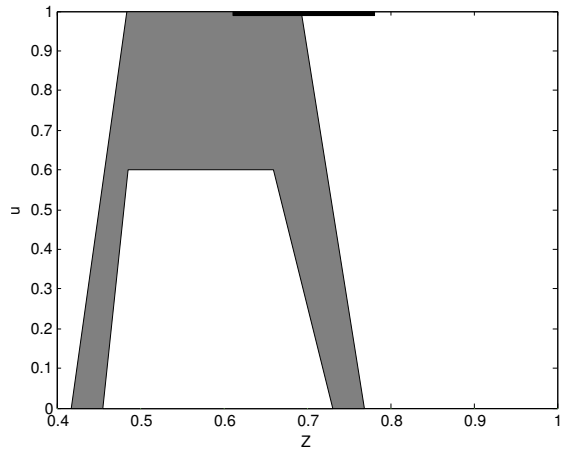
Overall, considering weight parameter in BK subproduct based inferencing gives direct impact in the inference results. This impact can be in any direction. Although there is no ground truth to compare with in this simulated case study, the results with weights considered are expected to be more reliable since more details, including the significance of each feature is considered.
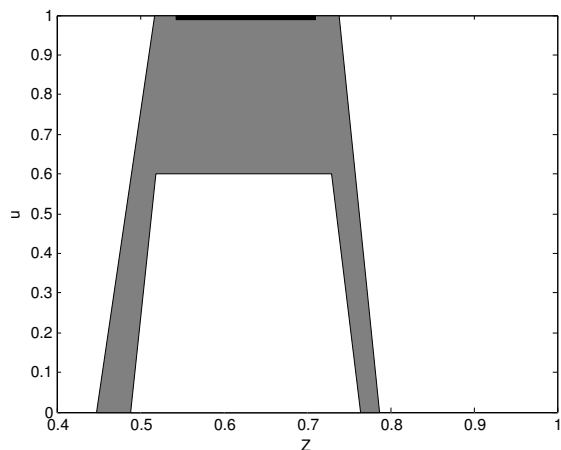
VI. CONCLUSION

Studies such as [1], [2], [33] have proved that BK subproduct has its advantages in formulating reasoning schemes. However, this does not means that the schemes developed from BK subproduct have no limitations. In this paper, we

(a) Inference result for patient a_1

(b) Inference result for patient a_2

(c) Inference result for patient a_3

Fig. 6. Final results of weighted inference are presented in term of trapezoidal membership functions. For comparisons, the results if grouping and weights are not considered (Table II) are presented as black bar in corresponding graphs.

first propose two enhancements on BK subproduct. These enhancements are: (i) interval type-2 fuzzy sets are used instead of ordinary type-1 fuzzy sets. This enhancement allows BK subproduct based inference engines to have better capability in handling uncertainties; and (ii) weight parameter is added to form weighted interval type-2 fuzzy based inference scheme. This enhancement allows the inference schemes to consider multiple groups of features at the same time, even though they have different levels of influence in the inference.

Due to BK subproduct is based on fuzzy subsethood measure, we also extended the implication operators based subsethood measure [4] to work with IT2FS. The outcome, however, does not have straight forward solution because the terms $\tilde{W}_m$ are IT2FS while $\tilde{R}_m \triangleleft \tilde{S}_m(a, c)$ are intervals. We adopted the solution of LWA [26], [27] by assuming that the intervals $\tilde{R}_m \triangleleft \tilde{S}_m(a, c)$ are special cases of IT2FS. Thus, the outcome can be solved easily by fining the FOU of $\tilde{R} \triangleleft \tilde{S}(a, c)$.

A simulated case study is provided to demonstrate the computing procedure of this weighted BK subproduct based inference engine. Our future work is looking forward to apply this into human motion analysis [34]–[36] and complex scene understanding context [37].

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