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Numerical solutions of fuzzy differential equations by an efficient Runge–Kutta method with generalized differentiability ☆

Ali Ahmadian [a], Soheil Salahshour [b], Chee Seng Chan [a,*], Dumitru Baleanu [c,d]

[a] *Centre of Image and Signal Processing, Faculty of Computer Science & Information Technology, University of Malaya, 50603 Kuala Lumpur, Malaysia*

[b] *Young Researchers and Elite Club, Mobarakeh Branch, Islamic Azad University, Mobarakeh, Iran*

[c] *Department of Mathematics, Çankaya University, 06530 Balgat, Ankara, Turkey*

[d] *Institute of Space Sciences, Magurele-Bucharest, Romania*

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Abstract

In this paper, an extended fourth-order Runge–Kutta method is studied to approximate the solutions of first-order fuzzy differential equations using a generalized characterization theorem. In this method, new parameters are utilized in order to enhance the order of accuracy of the solutions using evaluations of both f and f', instead of using the evaluations of f only. The proposed extended Runge–Kutta method and its error analysis, which guarantees pointwise convergence, are given in detail. Furthermore, the accuracy and efficiency of the proposed method are demonstrated in a series of numerical experiments.

Keywords: Fuzzy ordinary differential equations; Fuzzy differentiability; Characterization theorem; Error analysis; Runge–Kutta methods

1. Introduction

Fuzzy differential equations (FDEs) have been actively researched through the modeling of many exciting *real-world* problems in all branches of science and engineering, including hydraulic systems, population models, medicine, modeling of periodic phenomena through fuzzy systems etc. [20,37,13,12,40,43].

Initially, the concept of the fuzzy-valued function was introduced by Chang and Zadeh [29]. From this, Dubois and Prade [30] developed the approach by utilizing an extension principle. Thereafter, Puri and Ralescu [56] extended the concept of Hukuhara differentiability (H-differentiability) for set-valued functions to the class of fuzzy functions.

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* Corresponding author.
E-mail addresses: ahmadian.hosseini@um.edu.my (A. Ahmadian), soheilsalahshour@mau.ac.ir (S. Salahshour), cs.chan@um.edu.my (C.S. Chan), dumitru@cankaya.edu.tr (D. Baleanu).

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Thereafter, Kaleva [42] and Seikkala [59] used H-differentiability to develop some theorems of differential equations for FDEs.

In recent years, several pieces of research have been performed in order to investigate the theoretical foundations of FDEs in different classifications, such as the Cauchy problem of FDEs [60], fuzzy integro-differential equations [8], fuzzy relational equations [11], fuzzy functional differential equations [46], existence and uniqueness of solutions for fuzzy random differential equations [47], fuzzy stochastic differential equations [32] and fuzzy fractional differential equations [2,9]. Moreover, there have been many papers dedicated to the numerical solutions of FDEs [3,7,33], hybrid FDEs [6,54] and fuzzy fractional differential equations [4,5,58]. Specifically, Runge–Kutta (RK) methods were applied to solve FDEs under H-differentiability notion [1,34,52,53].

Unfortunately, the meaning of H-differentiability is defective such that the solutions of an FDE always have an increasing length of the support. This means that the diameter $diam\, x(t)$ of the solution is unbounded as $t \to \infty$ [35]. This circumstance leads to the conclusion that uncertainty of a fuzzy dynamical system increases chronologically. As a result, there has been immense efforts to establish the most suitable derivative for FDEs. Buckley and Feuring [21] applied Zadeh's extension principle for solving FDEs, but the lengths of its solution's support sets were also growing rapidly in most cases. This deficiency was solved by expounding an FDE as a set of differential inclusions [27,45]. Mizukoshi et al. [50] provided a comparison between finding the fuzzy solution of FDEs via Zadeh's extension principle and a family of differential inclusions. Our present research is not formed based on these concepts because the numerical work is not very well understood in these cases and an alien concept of derivative arises.

To overcome this, Bede and Gal [14] presented a concept of generalized H-differentiability of fuzzy-valued mappings which permits them to obtain the solutions of FDEs with a diminishing diameter of solutions values. This was followed up in the literature [15,16,19,23,25,28,48]. This comprehensive definition allows us to resolve the aforementioned disadvantages. Indeed, the strongly generalized derivative is defined for a larger class of fuzzy-number-valued functions than in the case of the Hukuhara derivative (H-derivative). In addition, some applications of generalized H-differentiability to the numerical solutions of FDEs were presented in [43,51]. Recently, Stefanini and Bede [61,62] with reference to the concept of generalization of the Hukuhara difference (H-difference) introduced generalized H-differentiability for interval valued functions. They concluded that this concept of differentiability has relationships with weakly generalized differentiability and strongly generalized H-differentiability. Furthermore, in [24], the authors studied the relationships between the strongly generalized H-differentiability and the generalized H-differentiability, showing the equivalence between these two concepts when the set of switching points of the interval valued function is finite. Therefore, to keep our task manageable, we confine it to considering the generalized H-differentiability, and provide an example with finite switching points to illustrate the later definition.

On the other hand, attempts have been made to reconstruct the classical RK methods due to the order of the method [31]. In addition, almost all RK methods were discussed in [22] – that the idea of these methods is to preserve the multi-stage nature of RK methods and permit more than one value to be passed from stage to stage. In addition, Phohomsiri and Udwadia [55] introduced the Accelerated RK method for solving ordinary differential equations (ODEs), which was studied completely in [63]. The nature of this method is to reuse previously calculated data and therefore it cannot be self-starting. In a different approach, Jackiewicz et al. [38] derived the two-step RK (TSRK) method. A more general category of these TSRK methods has been explained by Jackiewicz and Tracogna [39]. An alternative approach considered by Goeken et al. [36] proposed a class of RK methods using higher derivatives. In [64], the authors extended this technique to obtain the two step RK that can be regraded as the 'derivative-free' extended RK (ERK) method. Recently, Jikantoro et al. [41] derived the generalization of the ERK methods for solving non-autonomous ODEs. The ERK methods are more effective for cases where $f'(x, y)$ or y'' is of lower evaluation cost than f, which for a large class of fuzzy ordinary differential equations (FODEs) is very important. It is important to mention here that an obvious advantage of this technique is that only three evaluations of f are required per step, which reduces the computational cost dramatically. In contrast, arbitrary classical RK methods of order four would demand four evaluations of f per step. This partially motivates our interest, set out in this paper, to apply a family of ERK methods for solving FODEs.

The aim of this paper is to exploit ERK methods for solving non-autonomous systems as well as autonomous systems of FODEs under strongly generalized H-differentiability. We note that the application of ERK methods for solving autonomous FODEs has been introduced by Ghazanfari and Shakerami [34] under H-differentiability. However, most of the FODEs are expressed by non-autonomous systems. To this end, our proposed method uses the new terms of $k_i^{(j)}$ ($i = 1, .., 4$ and $j = 1, 2$) in order to derive the fourth-order ERK method for non-autonomous FODEs systems. Hence, it must handle a multitude of details – not only the use of first order derivatives f' and suitable error estimates, but also the selection of fuzzy differentiability type to ensure that the proposed formula can be employed in an adaptive style for the non-autonomous FODEs systems. To demonstrate the trustworthiness of the technique in the form of the adaptive version for both FODEs systems, we state and prove the convergence theory of the method. It is worth noting here that to solve numerical FODEs under fuzzy differentiability, we must solve two ODEs systems simultaneously in order to achieve the fuzzy approximate solution. This takes considerable computational cost. Thus, a numerical technique with low cost is highly desirable and is proposed in the current study as a way to overcome this deficiency in the literature.

Furthermore, another aspect that motivates this research is the numerical solution of FODEs under generalized H-differentiability. In this regard, we focus our interest in terms of generalized H-differentiability and analyze the results in comparison with H-differentiability for autonomous and non-autonomous systems of FODEs. Therewith, a characterization theorem presented by Bede [17] is used, which describes that a FDE under H-differentiability is equivalent to a system of ODEs under certain conditions that is appropriate for solving FDEs numerically. As a matter of fact, we exploit the extension of the characterization theorem which was introduced by Bede and Gal [18] to replace the FDE with its equivalent systems under generalized H-differentiability, and then solve two ODEs systems numerically by employing the proposed fourth-order ERK method.

The importance of this study, from a theoretical point of view, as well as from numerical applications, is that the present RK method is developed for solving a general class of FODEs under generalized H-differentiability. This can be of great help in the numerical study of FODEs. It is also worthwhile pointing out that the method has a lower computational cost and a more appropriate approximate fuzzy solution in comparison with the previous established papers [1,34]. It is based on the fact that the numerical results are built based on a well-defined combination of the generalized H-differentiability and a low cost fourth-order RK method. Moreover, we consider a fuzzy logistic equation that allows a switch between two concepts of fuzzy differentiability and solve it by using the fourth-order ERK method. To date, and to the best of our knowledge, this approach is limited and still traceless in the literature.

This paper is organized as follows: Section 2 contains preliminaries of the fuzzy number, fuzzy-valued function and H-differentiability. Additionally, the fourth-order ERK method is explained concisely under the non-fuzzy condition in this section. In Section 3, the fourth-order ERK method for solving FODEs is described under generalized H-differentiability, and the convergence conditions are provided. To demonstrate the validation and efficiency of the proposed method, a number of examples are solved. We give a brief summary of our results in Section 4. Finally, conclusions are drawn in Section 5.

2. Preliminaries

2.1. *Basic definitions*

In this section, the most important basic definitions of the notation used in fuzzy calculus and FODEs are recalled, as given in [33,49,56]. We consider $\mathbb{R}$ as the set of all real numbers. A fuzzy number is a mapping $u : \mathbb{R} \to [0, 1]$ with the following properties:
(a) u is upper semi-continuous,
(b) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y \geq min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0, 1]$,
(c) u is normal, i.e., $\exists\, x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,
(d) supp $u = \{x \in \mathbb{R} | u(x) > 0\}$ is the support of u, and its closure cl(supp u) is compact.

Let $\mathbb{E}$ be the set of all fuzzy numbers on $\mathbb{R}$. The r-level set of a fuzzy number $u \in \mathbb{E}$, $0 \leq r \leq 1$, denoted by $[u]_r$, is defined as

$$[u]_r = \begin{cases} \{x \in \mathbb{R} | u(x) \geq r\} & if\ 0 < r \leq 1 \\ cl(supp\, u) & if\ r = 0 \end{cases}$$

It is clear that the r-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(r), \bar{u}(r)]$, where $\underline{u}(r)$ denotes the left-hand endpoint of $[u]_r$. Since each $y \in \mathbb{R}$ can be regarded as a fuzzy number, $\widetilde{y}$ is defined by

$$\tilde{y}(t) = \begin{cases} 1 & if \quad t = y \\ 0 & if \quad t \neq y \end{cases}$$

For $u, v \in \mathbb{E}$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product $\lambda \odot u$ are defined by $[u + v]^r = [u]^r + [v]^r$, $[\lambda \odot u]^r = \lambda[u]^r$, $\forall r \in [0, 1]$, where $[u]^r + [v]^r$ means that the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^r$ results in the usual product between a scalar and a subset of $\mathbb{R}$.

The Hausdorff distance fuzzy numbers are given by $D : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}_+ \cup \{0\}$,

$$D(u, v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|\},$$

It is easy to see that D is a metric in $\mathbb{E}$ and has the following properties [56]
(i) $D(u \oplus w, v \oplus w) = D(u, v), \quad \forall u, v, w \in \mathbb{E}$,
(ii) $D(k \odot u, k \odot v) = |k|D(u, v), \quad \forall k \in \mathbb{R}, u, v \in \mathbb{E}$,
(iii) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e), \quad \forall u, v, w \in \mathbb{E}$,
(iv) $(\mathbb{E}, D)$ is a complete metric space.

Initially, the H-derivative for fuzzy mappings was introduced by Puri and Ralescu [56], and is based on the H-difference sets, as follows:

Definition 2.1. Let $x, y \in \mathbb{E}$. If there exists $z \in \mathbb{E}$ such that $x = y \oplus z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.

In this paper, the sign "$\ominus$" stands for H-difference. Also note that $x \ominus y \neq x + (-1)y$.

Definition 2.2. Let $f : \mathbb{R} \to \mathbb{E}$ be a fuzzy function. We say f is differentiable at $t_0 \in \mathbb{R}$, if there exists an element $f'(t_0) \in \mathbb{E}$ such that limits

$$\lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h}$$

exist and are equal to $f'(t_0)$. Here, the limits are taken in the metric space $(\mathbb{E}, D)$.

The above definition is a generalization of the H-differentiability of a set-valued function. From [35], it follows that a H-differentiable function has increasing length of support, so this definition of a derivative is very restrictive. In this regard, Bede and Gal [15] introduced a more generalized definition of H-differentiability which is our interest in this paper.

Definition 2.3. Let $f : (a, b) \to \mathbb{E}$ and $x_0 \in (a, b)$. We say that f is strongly generalized H-differentiable at x_0 if there exists an element $f'(x_0) \in \mathbb{E}$, such that
(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ and limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

or
(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0 - h) \ominus f(x_0)$ and limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or
(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0 - h) \ominus f(x_0)$ and limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or
(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0) \ominus f(x_0 - h)$ and limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

Bede [17] defined the **Characterization Theorem** that provides certain conditions under which a FDE is equivalent to a system of ODEs with respect to H-differentiability. Thereafter, Bede and Gal [18] proposed another version of this theorem for solving FDEs under generalized H-differentiability. Both of these characterization theorems are employed in the rest of this report to transfer the FODEs under fuzzy differentiability to ODEs systems.

As it is stated in [25,26,50,54], by Zadeh's extension principle, we obtain a fuzzy function generated from a non-fuzzy function $f(t, x(t))$ over $t \in [a, b]$. Hence, the fuzzy initial value problem (FIVP) can be presented as follows:

$$\begin{cases} x'(t) = f(t, x(t)), & t \in [a, b], \\ x(a) = x_0, \end{cases} \tag{2.1}$$

where $x_0 \in \mathbb{E}$ and $f : [a, b] \times \mathbb{E} \to \mathbb{E}$ is a fuzzy continuous function obtained by applying Zadeh's extension principle to the real function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$.

From the approach proposed in [57] we have:

$$[f(t, x(t))]^r = f(t, [\underline{x}_r, \bar{x}_r]) = [\min_{r\in[0,1]} f(t, [\underline{x}_r, \bar{x}_r]), \max_{r\in[0,1]} f(t, [\underline{x}_r, \bar{x}_r])],$$

or in an expanded parametric form, used throughout the paper, this can be presented by

$$[f(t, x(t))]^r = [\underline{f}^r(t, \underline{x}^r(t), \bar{x}^r(t)), \overline{f}^r(t, \underline{x}^r(t), \bar{x}^r(t))].$$

Definition 2.4. Let $f : (a, b) \to \mathbb{E}$. We say f is (i)-differentiable on (a, b) if f is differentiable with the meaning (i) of Definition 2.3 and similarly for (ii)-differentiability in Definition 2.3, case (ii).

Theorem 2.1. *([23]) Let $f : (a, b) \to \mathbb{E}$ be a function and denote $f(t) = (\underline{f}(t; r), \overline{f}(t; r))$, for each $r \in [0, 1]$. Then*
(1) If f is (i)-differentiable, then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ are differentiable functions and

$$f'(t) = (\underline{f}'(t; r), \overline{f}'(t; r)) \tag{2.2}$$

(2) If f is (ii)-differentiable, then $\underline{f}(t; r)$ and $\overline{f}(t; r)$ are differentiable functions and

$$f'(t) = (\overline{f}'(t; r), \underline{f}'(t; r)) \tag{2.3}$$

Definition 2.5. ([61]) We say that a point $t_0 \in (a, b)$ is a switching point for the differentiability of $f(t; r)$, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$, $\forall r \in [0, 1]$ such that

(type I) at $(t_1; r)$ (2.2) holds while (2.3) does not hold and at $(t_2; r)$ (2.3) holds and (2.2) does not hold, or

(type II) at $(t_1; r)$ (2.3) holds while (2.2) does not hold and at $(t_2; r)$ (2.2) holds and (2.3) does not hold.

By using Theorem 2.1, and the first type of characterization theorem [17], under certain conditions we may replace (2.1) by an equivalent ODEs system when $x(t)$ is considered as (i)-differentiable fuzzy-valued function:

$$\begin{cases} \underline{x}'(t; r) = \underline{f}^r(t, x) \equiv F(t, \underline{x}^r, \bar{x}^r), \underline{x}(a; r) = \underline{x_0}^r, \\ \bar{x}'(t; r) = \overline{f}^r(t, x) \equiv G(t, \underline{x}^r, \bar{x}^r), \bar{x}(a; r) = \overline{x_0}^r, \end{cases} \tag{2.4}$$

furthermore, by applying the second type of characterization theorem [18], FIVP (2.1) is equivalent with the union of the following two ODEs, if $x(t)$ is (ii)-differentiable fuzzy-valued function:

$$\begin{cases} (\underline{x}(t;r))' = \underline{f}^r(t, \underline{x}^r, \bar{x}^r), \\ (\bar{x}(t;r))' = \bar{f}^r(t, \underline{x}^r, \bar{x}^r), \\ \quad \underline{x}(a;r) = (\underline{x}_0^r), \\ \quad \bar{x}(a;r) = (\bar{x}_0^r), \end{cases}, r \in [0,1] \tag{2.5}$$

$$\begin{cases} (\underline{x}(t;r))' = \bar{f}^r(t, \underline{x}^r, \bar{x}^r), \\ (\bar{x}(t;r))' = \underline{f}^r(t, \underline{x}^r, \bar{x}^r), \\ \quad \underline{x}(a;r) = (\underline{x}_0^r), \\ \quad \bar{x}(a;r) = (\bar{x}_0^r), \end{cases}, r \in [0,1] \tag{2.6}$$

where $\bar{x}'(t,r)$ and $\underline{x}'(t,r)$ are variables for each $r \in [0,1]$.

2.2. *The fourth-order ERK method*

In this section, we explain the fourth-order ERK method in terms of crisp notion, which is introduced for autonomous ODEs systems in [64] and for non-autonomous ODEs in [41]. Firstly, the fourth-order ERK method for autonomous systems is discussed, then the proposed method is briefly described for non-autonomous systems.

Consider the autonomous IVP as:

$$\begin{cases} \frac{dy}{dx} = f(y), \\ y(a) = y_0, \quad x \in [a,b]. \end{cases} \tag{2.7}$$

We consider the fourth-order ERK formula which has the following form

$$y_{n+1} = y_n + h(b_1 k_1^{(1)} + b_2 k_2^{(1)} + b_3 k_3^{(1)}) + h^2(c_1 k_1^{(2)} + c_2 k_2^{(2)} + c_3 k_3^{(2)}), \tag{2.8}$$

where

$$\begin{aligned} k_1^{(1)} &= f(y_n), \quad k_2^{(1)} = f(y_n + h a_{21} k_1^{(1)}), \quad k_3^{(1)} = f(y_n + h a_{31} k_1^{(1)} + h a_{32} k_2^{(1)}), \\ k_1^{(2)} &= f'(y_n), \quad k_2^{(2)} = f'(y_n + h b_{21} k_1^{(1)}), \quad k_3^{(1)} = f'(y_n + h b_{31} k_1^{(1)} + h b_{32} k_2^{(1)}). \end{aligned} \tag{2.9}$$

The specific formula for autonomous ODEs systems after determining the coefficients of Equations (2.8) and (2.9) are as follows:

$$y_{n+1} = y_n + h f_n + \frac{1}{6} h^2 f_n' + \frac{1}{3} h^2 f_n'(y_n + \frac{1}{2} h f(y_n + \frac{1}{4} h f_n)),$$

or

$$y_{n+1} = y_n + h k_1^{(1)} + \frac{1}{6} h^2 k_1^{(2)} + \frac{1}{3} h^2 k_3^{(2)},$$

with local truncation error

$$T(t,h) = y(t_{n+1}) - y_{n+1} = \frac{h^5}{2880}(4 f_{yyyy} f_n^4 + 28 f_y f_{yyy} f_n^3 + 21 f_{yy}^2 f_n^3 + 69 f_y^2 f_{yy} f_n^2 + 24 f_y^4 f_n) + O(h^6),$$

where $f_n = f(y_n)$, $f_n' = f_y(y_n) f(y_n)$.

Remark 2.1. ([64]) The stability region of the formula (2.8) using two function evaluations of f' is similar to the classical fourth-order ERK method, since it has the same stability polynomials.

Let us consider the non-autonomous ODE system in the following form:

$$\begin{cases} \frac{dy}{dx} = f(x, y(x)), \\ y(a) = y_0, \quad x \in [a,b]. \end{cases} \tag{2.10}$$

The fourth-order ERK method for the problem (2.10) is determined as:

$$k_1^{(1)}=f(x_n,y_n),\ k_2^{(1)}=f(x_n+\tfrac{1}{2}h,y_n+\tfrac{h}{2}k_1^{(1)}),$$
$$k_3^{(1)}=f(x_n+h,y_n+hk_2^{(1)}),\ k_4^{(1)}=f(x_n+\tfrac{2}{5}h,y_n+\tfrac{7h}{25}k_1^{(1)}+\tfrac{2h}{25}k_2^{(1)}+\tfrac{h}{25}k_3^{(1)}), \tag{2.11}$$

and

$$k_1^{(2)}=f'(x_n,y_n),\ k_2^{(2)}=f'(x_n+\tfrac{1}{2}h,y_n+\tfrac{h}{2}k_1^{(1)}),$$
$$k_3^{(2)}=f'(x_n+h,y_n+hk_2^{(1)}),\ k_4^{(2)}=f'(x_n+\tfrac{2}{5}h,y_n+\tfrac{7h}{25}k_1^{(1)}+\tfrac{2h}{25}k_2^{(1)}+\tfrac{h}{25}k_3^{(1)}), \tag{2.12}$$

or in a concise form as:

$$\begin{aligned} y_{n+1} &= y_n+hf(x_n,y_n)+\tfrac{1}{8}h^2f'(x_n,y_n)+\tfrac{1}{36}h^2f'(x_n+h,y_n+hf(x_n+\tfrac{h}{2},y_n+\tfrac{h}{2}f(x_n,y_n)))\\ &\quad+\tfrac{25}{72}h^2f'(x_n+\tfrac{2h}{5},y_n+\tfrac{h}{25}(7f(x_n,y_n)+2f(x_n+\tfrac{h}{2},y_n+\tfrac{h}{2}f(x_n,y_n))\\ &\quad+f(x_n+h,y_n+hf(x_n+\tfrac{h}{2},y_n+\tfrac{h}{2}f(x_n,y_n)))))). \end{aligned} \tag{2.13}$$

Also, the local truncation error of the method is described by

$$\begin{aligned} LTE &= \tfrac{h^5}{720}\{f_y^4f+f_y^3f_x+f_{yy}f_yf_xf+f_{xy}f_yf_x-4(f_y^2f_{xx}+f_{xy}f_{xx}+f_{yy}^2f^3+ff_{yy}f_{xx})\\ &\quad-8f_{xy}^2f-12f_{xy}f_{yy}f^2-7f_{xy}f_y^2f-3f_{yy}f_y^2f^2\}+O(h^6). \end{aligned}$$

Remark 2.2. According to Lotkin [44], if the following bounds for f and its partial derivatives retain for $x\in[a,b]$ and $y\in[-\infty,\infty]$, we have

$$|f(x,y)|<Q,\quad |\frac{\partial^{i+j}f(x,y)}{\partial x^i\partial y^j}|<\frac{P^{i+j}}{Q^{j-1}},\quad i+j\le p, \tag{2.14}$$

where P and Q are positive constants and p is the order of the method. So for the proposed method described in Equation (2.13), we have $p=4$. Considering Eq. (2.13), we can gain an upper bound for the local truncation error as follows:

$$\left.\begin{aligned} |f_y^4f| &< Q(\tfrac{P^{0+1}}{Q^{1-1}})^4\\ |f_y^3f_x| &< (\tfrac{P^{0+1}}{Q^{1-1}})^3(\tfrac{P^{1+0}}{Q^{0-1}})\\ |f_{yy}f_yf_xf| &< Q(\tfrac{P^{1+1}}{Q^{2-1}})(\tfrac{P^{0+1}}{Q^{1-1}})(\tfrac{P^{1+0}}{Q^{0-1}})\\ |f_{xy}f_yf_x| &< (\tfrac{P^{1+1}}{Q^{1-1}})(\tfrac{P^{0+1}}{Q^{1-1}})(\tfrac{P^{1+0}}{Q^{0-1}})\\ |f_y^2f_{xx}| &< (\tfrac{P^{0+1}}{Q^{1-1}})^2(\tfrac{P^{1+1}}{Q^{0-1}})\\ &\vdots\\ f_{yy}f_y^2f^2 &< Q^2(\tfrac{P^{1+1}}{Q^{2-1}})(\tfrac{P^{0+1}}{Q^{1-1}})^2 \end{aligned}\right\} P^4Q\ldots \tag{2.15}$$

From the Eqs. (2.13)–(2.15), we acquire

$$|LTE|\le\frac{7}{120}P^4Qh^5. \tag{2.16}$$

3. The solution method

The contribution of this section is devoted to the generalization of the fourth-order ERK method for solving FODEs under generalized H-differentiability by employing the characterized theorems. Furthermore, the convergence theorem of the method is provided for non-autonomous and autonomous FODEs systems, respectively.

Now we consider the fuzzy approach of this method for solving *non-autonomous* FIVP (2.1). For a fixed r, in order to integrate the system given in (2.5) or (2.6) over the interval $[0,A]$, we discretize equally spaced grid points $0=t_0<t_1<\ldots<t_N=A$ where the exact solution at t_n is approximated by $y_{1n}(t_n;r)=[\underline{y_1}_n(t_n;r),\overline{y_1}_n(t_n;r)]$ such that

$$t_n = t_0 + nh, \quad h = \frac{A}{N}, \; 0 \le n \le N.$$

Denote the exact solution of system (2.5) at grid point t_n by $[\underline{Y_1}_n(t_n;r), \overline{Y_1}_n(t_n;r)]$ and similarly for the system (2.6) by $[\underline{Y_2}_n(t_n;r), \overline{Y_2}_n(t_n;r)]$. Assume that the approximate solutions are indicated at t_n by $y_{1n}(t_n;r) = [\underline{y_1}_n(t_n;r), \overline{y_1}_n(t_n;r)]$ and $y_{2n}(t_n;r) = [\underline{y_2}_n(t_n;r), \overline{y_2}_n(t_n;r)]$ under (i) and (ii)-differentiability, respectively.

The fourth-order ERK method under generalized H-differentiability for autonomous fuzzy systems is the fourth-order approximation of $\underline{Y_1}'_n(t_n;r)$, $\overline{Y_1}'_n(t_n;r)$,$\underline{Y_2}'_n(t_n;r)$, $\overline{Y_2}'_n(t_n;r)$ which can be written for the case of (i)-differentiability as:

$$\begin{cases} \underline{y_1}_{n+1}(t_{n+1};r) = \underline{y_1}_n(t_n;r) + h\underline{k_1}_1^{(1)} + \frac{h^2}{8}\underline{k_1}_1^{(2)} + \frac{h^2}{36}\underline{k_3}_1^{(2)} + \frac{25h^2}{72}\underline{k_4}_1^{(2)}, \\ \overline{y_1}_{n+1}(t_{n+1};r) = \overline{y_1}_n(t_n;r) + h\overline{k_1}_1^{(1)} + \frac{h^2}{8}\overline{k_1}_1^{(2)} + \frac{h^2}{36}\overline{k_3}_1^{(2)} + \frac{25h^2}{72}\overline{k_4}_1^{(2)}, \end{cases} \tag{3.17}$$

in which

$$\begin{cases} \underline{k_1}_1^{(1)} = \underline{f}(t_n, y_{1n}(t_n;r)), \overline{k_1}_1^{(1)} = \overline{f}(t_n, y_{1n}(t_n;r)), \\ \underline{k_1}_1^{(2)} = \underline{f'}(t_n, y_{1n}(t_n;r)), \overline{k_1}_1^{(2)} = \overline{f'}(t_n, y_{1n}(t_n;r)), \\ \underline{k_3}_1^{(2)} = \underline{f'}(t_n + h, y_{1n}(t_n;r) + hk_2^{(1)}), \overline{k_3}_1^{(2)} = \overline{f'}(t_n + h, y_{1n}(t_n;r) + hk_2^{(1)}), \\ \underline{k_4}_1^{(2)} = \underline{f'}(t_n + \frac{2h}{5}, y_{1n}(t_n;r) + \frac{7h}{25}k_1^{(1)} + \frac{2h}{25}k_2^{(1)} + \frac{h}{25}k_3^{(1)}), \\ \overline{k_4}_1^{(2)} = \overline{f'}(t_n + \frac{2h}{5}, y_{1n}(t_n;r) + \frac{7h}{25}k_1^{(1)} + \frac{2h}{25}k_2^{(1)} + \frac{h}{25}k_3^{(1)}), \end{cases} \tag{3.18}$$

and $k_1^{(1)} = [\underline{k_1}_1^{(1)}, \overline{k_1}_1^{(1)}]$, $k_2^{(1)} = [\underline{k_2}_1^{(1)}, \overline{k_2}_1^{(1)}]$, ..., $k_4^{(2)} = [\underline{k_4}_1^{(2)}, \overline{k_4}_1^{(2)}]$. It is worth noting that $f'(t_n, y_{1n}(t_n;r))$ is a partial derivative with respect to t that, in the expanded form, is

$$f'(t_n, y_{1n}(t;r)) = f_{t_n} + f_{y_{1n}} \frac{dy_{1n}(t_n;r)}{dt_n},$$

in which $y_{1n}(t_n;r) = [\underline{y_1}_n(t_n;r), \overline{y_1}_n(t_n;r)]$.

Also, in terms of (ii)-differentiability we obtain the following equations:

$$\begin{cases} \underline{y_2}_{n+1}(t_{n+1};r) = \underline{y_2}_n(t_n;r) + h\underline{k_1}_1^{(1)} + \frac{h^2}{8}\underline{k_1}_2^{(2)} + \frac{h^2}{36}\underline{k_3}_2^{(2)} + \frac{25h^2}{72}\underline{k_4}_2^{(2)}, \\ \overline{y_2}_{n+1}(t_{n+1};r) = \overline{y_2}_n(t_n;r) + h\overline{k_1}_1^{(1)} + \frac{h^2}{8}\overline{k_1}_2^{(2)} + \frac{h^2}{36}\overline{k_3}_2^{(2)} + \frac{25h^2}{72}\overline{k_4}_2^{(2)}, \end{cases} \tag{3.19}$$

where

$$\begin{cases} \underline{k_1}_2^{(1)} = \underline{f}(t_n, y_{2n}(t_n;r)), \overline{k_1}_2^{(1)} = \overline{f}(t_n, y_{2n}(t_n;r)), \\ \underline{k_1}_2^{(2)} = \overline{f'}(t_n, y_{2n}(t_n;r)), \overline{k_1}_2^{(2)} = \underline{f'}(t_n, y_{2n}(t_n;r)), \\ \underline{k_3}_2^{(2)} = \overline{f'}(t_n + h, y_{2n}(t_n;r) + hk_2^{(1)}), \overline{k_3}_2^{(2)} = \underline{f'}(t_n + h, y_{2n}(t_n;r) + hk_2^{(1)}), \\ \underline{k_4}_2^{(2)} = \overline{f'}(t_n + \frac{2h}{5}, y_{2n}(t_n;r) + \frac{7h}{25}k_1^{(1)} + \frac{2h}{25}k_2^{(1)} + \frac{h}{25}k_3^{(1)}), \\ \overline{k_4}_2^{(2)} = \underline{f'}(t_n + \frac{2h}{5}, y_{2n}(t_n;r) + \frac{7h}{25}k_1^{(1)} + \frac{2h}{25}k_2^{(1)} + \frac{h}{25}k_3^{(1)}), \end{cases} \tag{3.20}$$

and $\underline{y_1}_0(r) = \underline{x}(0;r)$, $\overline{y_1}_0(r) = \bar{x}(0;r)$, $\underline{y_2}_0(r) = \underline{x}(0;r)$, $\overline{y_2}_0(r) = \bar{x}(0;r)$. Hence, (3.17) and (3.19) represent the approximation of $Y_1(r)$ and $Y_2(r)$. Let $r \in [0, 1]$. For the convergence of (3.17) and (3.19) the following relations hold:

$$\begin{cases} \lim_{h\to 0} \underline{y_1}_n(t_n;r) = \underline{x}(t;r); \; \lim_{h\to 0} \overline{y_1}_n(t_n;r) = \bar{x}(t;r), \\ \lim_{h\to 0} \underline{y_2}_n(t_n;r) = \underline{x}(t;r); \; \lim_{h\to 0} \overline{y_2}_n(t_n;r) = \bar{x}(t;r), \end{cases}$$

which is an application of Theorem 1 in Bede [17], Theorem 4.3 in [51] and Lemmas 1 and 2 in [49] as follows.

Let $F(u, v)$ and $G(u, v)$ be the functions F and G of Eqs. (2.5) and (2.6) where u and v are constants and $u \le v$. The domain where F and G are defined is given by

$$K = \{(u, v) | -\infty < v < \infty, -\infty < u \le v\}.$$

Theorem 3.1. *Consider the system (2.5) and (3.17) or system (2.6) and (3.19),* $r \in [0, 1]$

$$\begin{cases} \lim_{h\to 0} \underline{y_1}_n(t_n; r) = \underline{x}(t; r); \ \lim_{h\to 0} \overline{y_1}_n(t_n; r) = \bar{x}(t; r), \\ \lim_{h\to 0} \underline{y_2}_n(t_n; r) = \underline{x}(t; r); \ \lim_{h\to 0} \overline{y_2}_n(t_n; r) = \bar{x}(t; r), \end{cases} \tag{3.21}$$

provided that in each step, $y_{1n}(t_n; r) = [\underline{y_1}_n(t_n; r), \overline{y_1}_n(t_n; r)]$ *and* $y_{2n}(t_n; r) = [\underline{y_2}_n(t_n; r), \overline{y_2}_n(t_n; r)]$ *define a fuzzy-valued function. Additionally, the conditions stated in the characterization theorems hold for* f*.*

Proof. By considering the approximate solution (3.21), it is sufficient to show

$$\begin{aligned} &\lim_{h\to 0} \underline{y_1}_n(t_n; r) = \underline{x}(t; r), \\ &\lim_{h\to 0} \overline{y_1}_n(t_n; r) = \bar{x}(t; r). \end{aligned} \tag{3.22}$$

where $t_N = T$. For $n = 1, ..., N-1$, by using the exact solution, the following outcome will be obtained:

$$\begin{aligned} &\underline{x}(t_{n+1}; r) = \underline{x}(t_n; r) + F(\underline{x}(t_n; r), \bar{x}(t_n; r)) + \tfrac{7h^5}{120} P^4 Q + O(h^6), \\ &\bar{x}(t_{n+1}; r) = \bar{x}(t_n; r) + G(\underline{x}(t_n; r), \bar{x}(t_n; r)) + \tfrac{7h^5}{120} P^4 Q + O(h^6). \end{aligned} \tag{3.23}$$

Denote

$$W_n = \underline{x}(t_n; r) - \underline{y_1}_n(t_n; r), \quad V_n = \bar{x}(t_n; r) - \overline{y_1}_n(t_n; r).$$

Therefore, from Eqs. (3.17) and (3.23), we achieve

$$\begin{aligned} &W_{n+1} = W_n + F(\underline{x}(t_n; r), \bar{x}(t_n; r)) - F(\underline{y_1}_n(t_n; r), \overline{y_1}_n(t_n; r)) + \tfrac{7h^5}{120} P^4 Q + O(h^6), \\ &V_{n+1} = V_n + G(\underline{x}(t_n; r), \bar{x}(t_n; r)) - G(\underline{y_1}_n(t_n; r), \overline{y_1}_n(t_n; r)) + \tfrac{7h^5}{120} P^4 Q + O(h^6). \end{aligned}$$

Then, we have

$$\begin{aligned} &|W_{n+1}| \le |W_n| + 2L_1 h \max\{|W_n|, |V_n|\} + \tfrac{7h^5}{120} P^4 Q + O(h^6), \\ &|W_{n+1}| \le |V_n| + 2L_2 h \max\{|V_n|, |V_n|\} + \tfrac{7h^5}{120} P^4 Q + O(h^6), \end{aligned}$$

and assuming $L = \max\{L_1, L_2\}$,

$$\begin{aligned} &|W_{n+1}| \le |W_n| + 2Lh \max\{|W_n|, |V_n|\} + \tfrac{7h^5}{120} P^4 Q + O(h^6), \\ &|W_{n+1}| \le |V_n| + 2Lh \max\{|V_n|, |V_n|\} + \tfrac{7h^5}{120} P^4 Q + O(h^6), \end{aligned}$$

are derived, where $|U_n| = |W_n| + |V_n|$. Hence, by Lemma 2 in [49], without loss of generality, we have that

$$|U_n| \le (1 + 4Lh)^N |U_1| + (\frac{7h^5}{120} P^4 Q + O(h^6)) \frac{(1 + 4Lh)^n - 1}{4Lh}.$$

Since $W_1 \approx 0$, $V_1 \approx 0$

$$|U_n| \le (\frac{7h^5}{120} P^4 Q) \frac{e^{4LT} - 1}{L} + O(h^6),$$

and if $h \to 0$, we get to $W_N \to 0$, $V_N \to 0$ which completes the proof. □

Remark 3.1. The proof of Theorem 3.1 for the approximate solution (3.19) under (ii)-differentiability is quite similar to the demonstration of the proof under the (i)-differentiability assumption.

In a similar way, we describe the fourth-order ERK formula for solving *autonomous* FODEs. This method has been applied under H-differentiability in [34]. Here, the method is developed under generalized H-differentiability by considering the generalized characterization theorem [18].

Let us consider the autonomous FODE as follows:

$$\begin{cases} y'(t) = f(y(t)), \\ y(0) = y_0 \quad t \in I = [0, T],\, y_0 \in \mathbb{E}. \end{cases} \tag{3.24}$$

The fourth-order ERK method can be formulated with respect to (3.24) as follows:
case (i)-differentiability as:

$$\begin{cases} \underline{y_1}_{n+1}(t_{n+1}; r) = \underline{y_1}_n(t_n; r) + h\underline{k_1}_1^{(1)} + \frac{h^2}{6}\underline{k_1}_1^{(2)} + \frac{h^2}{3}\underline{k_3}_1^{(2)}, \\ \overline{y_1}_{n+1}(t_{n+1}; r) = \overline{y_1}_n(t_n; r) + h\overline{k_1}_1^{(1)} + \frac{h^2}{6}\overline{k_1}_1^{(2)} + \frac{h^2}{3}\overline{k_3}_1^{(2)}, \end{cases} \tag{3.25}$$

and for case (ii)-differentiability, we reach the following equations:

$$\begin{cases} \underline{y_2}_{n+1}(t_{n+1}; r) = \underline{y_2}_n(t_n; r) + h\underline{k_1}_2^{(1)} + \frac{h^2}{6}\underline{k_1}_2^{(2)} + \frac{h^2}{3}\underline{k_3}_2^{(2)}, \\ \overline{y_2}_{n+1}(t_{n+1}; r) = \overline{y_2}_n(t_n; r) + h\overline{k_1}_2^{(1)} + \frac{h^2}{6}\overline{k_1}_2^{(2)} + \frac{h^2}{3}\overline{k_3}_2^{(2)}, \end{cases} \tag{3.26}$$

where $k_1{}_1^{(1)}, k_1{}_1^{(2)}, k_3{}_1^{(2)}, k_1{}_2^{(1)}, k_1{}_2^{(2)}, k_3{}_2^{(2)}$ for (i) and (ii)-differentiability, respectively, are as follows:

$$\begin{cases} \underline{k_1}_1^{(1)} = \underline{f}(y_{1n}(t_n; r)),\, \overline{k_1}_1^{(1)} = \overline{f}(y_{1n}(t_n; r)), \\ \underline{k_1}_1^{(2)} = \underline{f}'(y_{1n}(t_n; r)),\, \overline{k_1}_1^{(2)} = \overline{f}'(y_{1n}(t_n; r)), \\ \underline{k_3}_1^{(2)} = \underline{f}'(y_{1n}(t_n; r) + \frac{1}{2}h\ f(y_{1n}(t_n; r) + \frac{1}{4}hf(y_{1n}(t_n; r)))), \\ \overline{k_3}_1^{(2)} = \overline{f}'(y_{1n}(t_n; r) + \frac{1}{2}h\ f(y_{1n}(t; r) + \frac{1}{4}hf(y_{1n}(t_n; r)))), \end{cases} \tag{3.27}$$

and

$$\begin{cases} \underline{k_1}_2^{(1)} = \underline{f}(y_{2n}(t_n; r)),\, \overline{k_1}_2^{(1)} = \overline{f}(y_{2n}(t_n; r)), \\ \underline{k_1}_2^{(2)} = \overline{f}'(y_{2n}(t_n; r)),\, \overline{k_1}_2^{(2)} = \underline{f}'(y_{2n}(t_n; r)), \\ \underline{k_3}_2^{(2)} = \overline{f}'(y_{2n}(t_n; r) + \frac{1}{2}h\ f(y_{2n}(t_n; r) + \frac{1}{4}hf(y_{2n}(t_n; r)))), \\ \overline{k_3}_2^{(2)} = \underline{f}'(y_{2n}(t_n; r) + \frac{1}{2}h\ f(y_{2n}(t_n; r) + \frac{1}{4}hf(y_{2n}(t_n; r)))), \end{cases} \tag{3.28}$$

in which $f'(y_{1n}(t_n; r))$ and $f'(y_{2n}(t_n; r))$ are partial derivatives with respect to t that can be written in the expanded form as:

$$\underline{f}'(\underline{y_1}_n(t_n, r), \overline{y_1}_n(t_n, r)) = \min_{u \in [\underline{y_1}_n(t_n, r), \overline{y_1}_n(t_n, r)]} \{f_u.\frac{du}{dt_n} | r \in [0, 1]\},$$

$$\overline{f}'(\underline{y_1}_n(t_n, r), \overline{y_1}_n(t_n, r)) = \max_{u \in [\underline{y_1}_n(t_n, r), \overline{y_1}_n(t_n, r)]} \{f_u.\frac{du}{dt_n} | r \in [0, 1]\},$$

and in a similar way, it can be written for $f'(y_{2n}(t_n; r))$.

Theorem 3.2. *Consider the system (2.5) and (3.25) or system (2.6) and (3.26), $r \in [0, 1]$*

$$\begin{cases} \lim_{h \to 0} \underline{y_1}_n(t_n; r) = \underline{x}(t; r);\ \lim_{h \to 0} \overline{y_1}_n(t_n; r) = \bar{x}(t; r), \\ \lim_{h \to 0} \underline{y_2}_n(t_n; r) = \underline{x}(t; r);\ \lim_{h \to 0} \overline{y_2}_n(t_n; r) = \bar{x}(t; r), \end{cases} \tag{3.29}$$

provided that in each step, $y_{1n}(t_n; r) = [\underline{y_1}_n(t_n; r), \overline{y_1}_n(t_n; r)]$ and $y_{2n}(t_n; r) = [\underline{y_2}_n(t_n; r), \overline{y_2}_n(t_n; r)]$ define a fuzzy-valued function. Also, the conditions stated in the characterization theorems hold for f.

Proof. If system (2.5) and (3.25) or system (2.6) and (3.26) are considered, then we can prove the convergence of the method according to the demonstration of Theorem 3.1 and Theorem 4.3 in [34]. □

4. Numerical examples

In this section, a number of numerical examples are given to demonstrate the effectiveness of the proposed method. The method is compared with the fourth-order classical ERK method under (i) and (ii)-differentiability. Moreover, the absolute error function, which is the difference between the fuzzy approximate solutions $[y_N]^r = [\underline{y}_N^r, \overline{y}_N^r]$ and the corresponding exact solutions $y(t;r) = [\underline{y}(t;r), \overline{y}(t;r)]$ i.e. $[N_e]^r = [\underline{N}_e^r, \overline{N}_e^r] = \left[|\underline{y}_n^r - \underline{y}^r|, |\overline{y}_n^r - \overline{y}^r|\right]$, is provided for each of examples.

Remark 4.1. As it is stated in Theorem 2.5 in [18], the solution of FDEs is not unique. Although this may appear to be a disadvantage, this deficiency can be used as an advantage, since we may sometimes have the opportunity to choose the solution which is closer to the real system, and better reflects the behavior of the system. This advantage is shown by the following example.

Example 4.1. We consider the following FODE [52]:

$$y'(t) = \bar{c}y(t), \qquad y(0) = y_0, \tag{4.30}$$

where $y(t)$ and $\bar{c} = (-4/-3/-2)$ are fuzzy numbers. Also let $I = [0, 1]$ and $y(0;r) = [8 + 0.5r, 9 - 0.5r]$.

The analytical solution under (i)-differentiability is

$$\underline{Y_1}(t;r) = \frac{1}{2}\left[(8+0.5r) - (9-0.5r)\sqrt{\frac{4-r}{2+r}}\right]e^{\sqrt{(4-r)(2+r)}t} + \frac{1}{2}\left[(8+0.5r) + (9-0.5r)\sqrt{\frac{4-r}{2+r}}\right]e^{-\sqrt{(4-r)(2+r)}t}$$
$$\overline{Y_1}(t;r) = \frac{1}{2}\left[(9-0.5r) - (8+0.5r)\sqrt{\frac{2+r}{4-r}}\right]e^{\sqrt{(4-r)(2+r)}t} + \frac{1}{2}\left[(9-0.5r) + (8+0.5r)\sqrt{\frac{2+r}{4-r}}\right]e^{-\sqrt{(4-r)(2+r)}t}$$

The (ii)-solution is given by

$$\underline{Y_2}(t;r) = (8+0.5r)e^{-(2+r)t},$$
$$\overline{Y_2}(t;r) = (9-0.5r)e^{-(4-r)t}.$$

The approximate solution under (i)-differentiability by using Eqs. (3.25) and (3.27) is as follows:

$$\begin{cases} \underline{k_1}_1^{(1)} = \underline{f}(y_{1n}^r) = -(4-r)\overline{y_1}_n^r, \\ \overline{k_1}_1^{(1)} = \overline{f}(y_{1n}^r) = -(2+r)\underline{y_1}_n^r, \\ \underline{k_1}_1^{(2)} = \underline{f}'(y_{1n}(t_n;r)) = (4-r)(2+r)\underline{y_1}_n^r, \\ \overline{k_1}_1^{(2)} = \overline{f}'(y_{1n}(t_n;r)) = (4-r)(2+r)\overline{y_1}_n^r, \end{cases}$$

and

$$\begin{cases} \underline{k_3}_1^{(2)} = \underline{f}'(y_{1n}(t_n;r) + \frac{1}{2}h\ f(y_{1n}(t_n;r) + \frac{1}{4}hf(y_{1n}(t_n;r)))) \\ \quad = (4-r)(2+r)[\underline{y_1}_n^r - \frac{1}{2}h(4-r)\overline{y_1}_n^r + \frac{1}{8}h(4-r)(2+r)\underline{y_1}_n^r], \\ \overline{k_3}_1^{(2)} = \overline{f}'(y_{1n}(t_n;r) + \frac{1}{2}h\ f(y_{1n}(t_n;r) + \frac{1}{4}hf(y_{1n}(t_n;r)))) \\ \quad = (4-r)(2+r)[\overline{y_1}_n^r - \frac{1}{2}h(2+r)\underline{y_1}_n^r + \frac{1}{8}h(4-r)(2+r)\overline{y_1}_n^r]. \end{cases}$$

Now, by employing formula (3.25) we have:

$$\begin{cases} \underline{y_1}_{n+1}^r = [1 + \frac{h^2}{2}(4-r)(2+r) + \frac{h^4}{24}(4-r)^2(2+r)^2]\underline{y_1}_n^r - [h(4-r) + \frac{h^3}{6}(4-r)^2(2+r)]\overline{y_1}_n^r, \\ \overline{y_1}_{n+1}^r = [1 + \frac{h^2}{2}(2+r)(4-r) + \frac{h^4}{24}(2+r)^2(4-r)^2]\overline{y_1}_n^r - [h(2+r) + \frac{h^3}{6}(2+r)^2(4-r)]\underline{y_1}_n^r, \\ \underline{y_1}_0^r = \underline{x}_0^r, \\ \overline{y_1}_0^r = \overline{x}_0^r, \end{cases}$$

Table 1
Approximate solution of the fourth-order ERK method (y_1^r), exact solution (Y_1^r) and absolute error function (N_e^r) under (i)-differentiability at $t = 1$ for $r \in [0, 1]$, Example 4.1.

r	$\underline{y_1}^r$	$\underline{Y_1}^r$	$\underline{N}_e^r$	$\overline{y_1}^r$	$\overline{Y_1}^r$	$\overline{N}_e^r$
0	−3.937800e1	−3.938288e1	4.887053e−3	2.871091e1	2.871421e1	3.290109e−3
0.1	−3.568715e1	−3.569184e1	4.692112e−3	2.703668e1	2.703995e1	3.270464e−3
0.2	−3.185046e1	−3.185487e1	4.409574e−3	2.507075e1	2.507393e1	3.175881e−3
0.3	−2.789779e1	−2.790183e1	4.045308e−3	2.282183e1	2.282483e1	3.003787e−3
0.4	−2.385973e1	−2.386334e1	3.607525e−3	2.030125e1	2.030401e1	2.753871e−3
0.5	−1.976738e1	−1.977048e1	3.106500e−3	1.752300e1	1.752543e1	2.428150e−3
0.6	−1.565199e1	−1.565455e1	2.554228e−3	1.450356e1	1.450558e1	2.030975e−3
0.7	−1.154477e1	−1.154674e1	1.964035e−3	1.126176e1	1.126333e1	1.568946e−3
0.8	−7.476534e0	−7.477884e0	1.350158e−3	7.818726e0	7.819776e0	1.050765e−3
0.9	−3.477425e0	−3.478152e0	7.272931e−4	4.197568e0	4.198055e0	4.870125e−4
1	4.233002e−1	4.231901e−1	1.101453e−4	4.233002e−1	4.231901e−1	1.101453e−4

in which the interval is divided into $N = 10$ equally spaced subintervals. It is necessary to mention here that $f'(\underline{y}_{1n}^{\ r}, \overline{y}_{1n}^{\ r})$ is a partial derivative with respect to t.

Similarly, using Eqs. (3.26) and (3.28), the approximate solution of the problem (4.30) under (ii)-differentiability is given by:

$$\begin{cases} \underline{k_1}_2^{(1)} = \underline{f}(y_{2n}^r) = -(2+r)\underline{y_2}_n^r, \\ \overline{k_1}_2^{(1)} = \overline{f}(y_{2n}^r) = -(4-r)\overline{y_2}_n^r, \\ \underline{k_1}_2^{(2)} = \overline{f}'(y_{2n}(t_n; r)) = (2+r)^2\underline{y_2}_n^r, \\ \overline{k_1}_2^{(2)} = \underline{f}'(y_{2n}(t_n; r)) = (4-r)^2\overline{y_2}_n^r, \end{cases}$$

and

$$\begin{cases} \underline{k_3}_2^{(2)} = \overline{f}'(y_{2n}(t_n; r) + \frac{1}{2}h\ f(y_{2n}(t_n; r) + \frac{1}{4}hf(y_{2n}(t_n; r)))) \\ \qquad = (2+r)^2\underline{y_2}_n^r[1 - \frac{1}{2}h(2+r) + \frac{1}{8}h(2+r)^2], \\ \overline{k_3}_2^{(2)} = \underline{f}'(y_{2n}(t_n; r) + \frac{1}{2}h\ f(y_{2n}(t_n; r) + \frac{1}{4}hf(y_{2n}(t_n; r)))) \\ \qquad = (4-r)^2\overline{y_2}_n^r[1 - \frac{1}{2}h(4-r) + \frac{1}{8}h(4-r)^2]. \end{cases}$$

By exploiting formula (3.26) we have:

$$\begin{cases} \underline{y_2}_{n+1}^r = [1 - h(2+r) + \frac{h^2}{2}(2+r)^2 - \frac{h^3}{6}(2+r)^3 + \frac{h^4}{24}(2+r)^4]\underline{y_2}_n^r, \\ \overline{y_2}_{n+1}^r = [1 - h(4-r) + \frac{h^2}{2}(4-r)^2 - \frac{h^3}{6}(4-r)^3 + \frac{h^4}{24}(4-r)^4]\overline{y_2}_n^r, \\ \underline{y_2}_0^r = \underline{x}_0^r, \\ \overline{y_2}_0^r = \overline{x}_0^r, \end{cases}$$

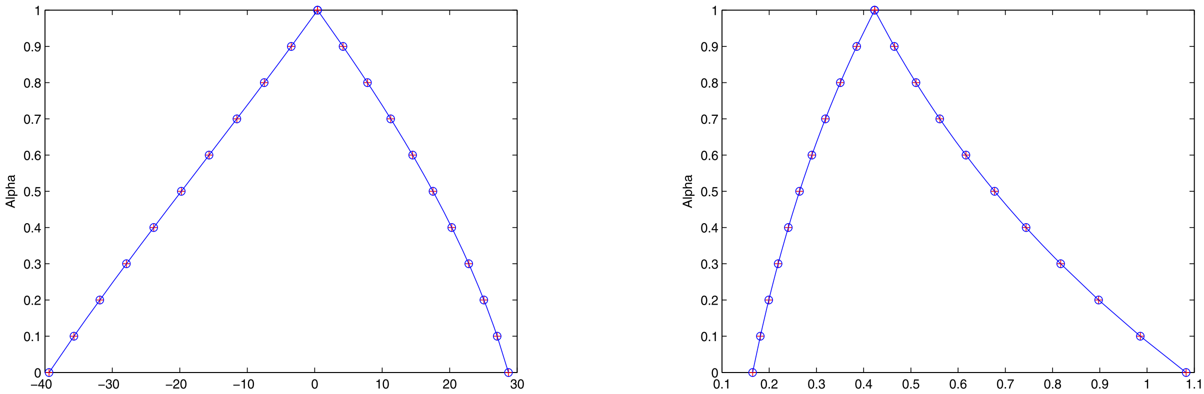
in which the interval is divided into $N = 10$ equally spaced subintervals. $f'(\underline{y}_{1n}^{\ r}, \overline{y}_{1n}^{\ r})$ is a partial derivative with respect to t.

The (i) and (ii)-exact and approximate solutions, given by the fourth-order ERK method, are shown in Tables 1 and 2 at $t = 1$. Alongside this, the absolute error functions (N_e^r) are presented to demonstrate the high accuracy of the proposed technique under both types of fuzzy differentiability. Fig. 1 displays the fuzzy approximation of the fourth-order ERK method and the exact solution under both types of fuzzy differentiability at $t = 1$ for $r \in [0, 1]$. It is clear that the approximate solution is in excellent agreement with the exact solution. Moreover, the width of the fuzzy solution under (i)-differentiability is considerably greater than the other.

Example 4.2. Let us consider an electrical circuit (RL circuit) with an AC source. As can be seen in Fig. 2, RL circuits are circuits that contain both an inductor (L) and a resistor (R). The resistor and inductor are in series with one another.

Table 2
Approximate solution of the fourth-order ERK method ($\underline{y_2}^r$), exact solution ($\underline{Y_2}^r$) and absolute error function ($\underline{N}_e^r$) under (ii)-differentiability at $t = 1$ for $r \in [0, 1]$, Example 4.1.

r	$\underline{y_2}^r$	$\underline{Y_2}^r$	$\underline{N}_e^r$	$\overline{y_2}^r$	$\overline{Y_2}^r$	$\overline{N}_e^r$
0	1.082716e0	1.082682e0	3.412155e−5	1.650374e−1	0.164840e0	1.967231e−4
0.1	9.858142e−1	9.857742e−1	3.998434e−5	1.813540e−1	0.181165e0	1.888891e−4
0.2	8.975519e−1	8.975055e−1	4.632400e−5	1.992806e−1	0.199099e0	1.807677e−4
0.3	8.171626e−1	8.171096e−1	5.311482e−5	2.189756e−1	0.218803e0	1.723932e−4
0.4	7.439475e−1	7.438872e−1	6.032561e−5	2.406125e−1	0.240449e0	1.638032e−4
0.5	6.772691e−1	6.772012e−1	6.792032e−5	2.643821e−1	0.264227e0	1.550379e−4
0.6	6.165465e−1	6.164706e−1	7.585884e−5	2.904936e−1	0.290347e0	1.461398e−4
0.7	5.612501e−1	5.611660e−1	8.409766e−5	3.191766e−1	0.319039e0	1.371539e−4
0.8	5.108971e−1	5.108045e−1	9.259063e−5	3.506831e−1	0.350554e0	1.281272e−4
0.9	4.650474e−1	4.649462e−1	1.012896e−4	3.852900e−1	0.385170e0	1.191079e−4
1	4.233002e−1	4.231900e−1	1.101453e−4	4.233002e−1	0.423190e0	1.101453e−4



(a) Fuzzy approximate and exact solutions under (i)- differentiability

(b) Fuzzy approximate and exact solutions under (ii)- differentiability

Fig. 1. A comparison between the fuzzy approximate solution by the fourth-order ERK method (+) and fuzzy exact solution (∘) at $t = 1$ for $r \in [0, 1]$, Example 4.1.

There is also a battery with emf ϵ, and a switch that is initially in the "no battery" position. At first, there is no current in the circuit. Due to the response of the inductor to change, it opposes any change in current = in an RL circuit as time goes by. The potential differences across the resistor and inductor also change, but the loop rule is satisfied at all times [65].

However, environmental conditions, unknown effects over the circuit and vague values of elements are the main factors of uncertain variables in circuit analysis. In fact, the current of a circuit depends on applied voltage to the circuit, and impedance of the circuit. If the applied voltage is not constant, and there are differences with theoretical aspects then uncertainty exists between the theoretical response and practical results [66,67]. Hence, the fuzzy concept helps us to interpret the problem as follows:

$$\begin{cases} y'(t) = -\frac{R}{L}y(t) + v(t), & t \in [0, 1], \\ y(0; r) = (0.96 + 0.04r, 1.01 - 0.01r), \end{cases} \tag{4.31}$$

where $r \in [0, 1]$ and R is the circuit resistance and L is a coefficient corresponding to the solenoid.

Assume that $v(t) = \sin(t)$, $R = 1\Omega$ and $L = 1\,\text{H}$. Therefore (4.31) can be rewritten as:

$$\begin{cases} y'(t) = -y(t) + \sin(t), & t \in [0, 1], \\ y(0; r) = (0.96 + 0.04r, 1.01 - 0.01r), \end{cases} \tag{4.32}$$

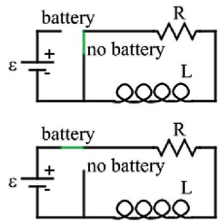


Fig. 2. An LR circuit with a battery, resistor, inductor, and switch, Example 4.2.

and the exact solution of this problem under (ii)-differentiability is:

$$\begin{cases} \underline{Y}_1(t;r) = \frac{1}{2}(\sin(t) - \cos(t)) + e^{-t}(1.46 + 0.04\mathrm{r}), \\ \overline{Y}_1(t;r) = \frac{1}{2}(\sin(t) - \cos(t)) + e^{-t}(1.51 - 0.01\mathrm{r}), \end{cases}$$

and by using (i)-differentiability, the analytical solution is given by:

$$\begin{cases} \underline{Y}_1(t;r) = -\frac{1}{2}(0.05 - 0.05r)e^{t} + \frac{1}{2}(2.97 + 0.03r)e^{-t} + \frac{1}{2}(\sin(t) - \cos(t)), \\ \overline{Y}_1(t;r) = \frac{1}{2}(0.05 - 0.05r)e^{t} + \frac{1}{2}(2.97 + 0.03r)e^{-t} + \frac{1}{2}(\sin(t) - \cos(t)). \end{cases}$$

It should be noted that $f'(t, y(t;r)) = [\underline{y}(t;r) + \cos(t) - \sin(t), \bar{y}(t;r) + \cos(t) - \sin(t)]$. By applying the technique described in Section 3, Eqs. (3.17) and (3.18), for non-autonomous FODE under (i)-differentiability we have:

$$\begin{cases} \underline{k1}_1^{(1)} = (-\bar{y}1_n^r + \sin(t_n)); \quad \overline{k1}_1^{(1)} = (-\underline{y}_{1n}^{\,r} + \sin(t_n)), \\ \underline{k1}_1^{(2)} = (\underline{y}_{1n}^{\,r} + \cos(t_n) - \sin(t_n)); \quad \overline{k1}_1^{(2)} = (\bar{y}1_n^r + \cos(t_n) - \sin(t_n)). \end{cases} \tag{4.33}$$

Moreover, we can obtain

$$\begin{cases} \underline{k3}_1^{(2)} = (1 + \frac{h^2}{2})\underline{y}_{1n}^{\,r} - h\bar{y}1_n^r - \frac{h^2}{2}\sin(t_n) + (h-1)\sin(t_n + h) + \cos(t_n + h), \\ \overline{k3}_1^{(2)} = (1 + \frac{h^2}{2})\bar{y}1_n^r - h\underline{y}_{1n}^{\,r} - \frac{h^2}{2}\sin(t_n) + (h-1)\sin(t_n + h) + \cos(t_n + h), \end{cases} \tag{4.34}$$

and

$$\begin{cases} \underline{k4}_1^{(2)} = \underline{y}_{1n}^{\,r} + \frac{h^2}{2}((-10 - \frac{h^2}{2})\bar{y}1_n^r + 2h\underline{y}_{1n}^{\,r} + (7 - h + \frac{h^2}{2})\sin(t_n) + (2-h)\sin(t_n + \frac{h}{2}) + \sin(t_n + h)), \\ \overline{k4}_1^{(2)} = \bar{y}1_n^r + \frac{h^2}{2}((-10 - \frac{h^2}{2})\underline{y}_{1n}^{\,r} + 2h\bar{y}1_n^r + (7 - h + \frac{h^2}{2})\sin(t_n) + (2-h)\sin(t_n + \frac{h}{2}) + \sin(t_n + h)), \end{cases} \tag{4.35}$$

i.e. the approximate solution $y1_{n+1}^r$ is achieved immediately from formula (3.17) by using Eqs. (4.33)–(4.35).

We can reach the fuzzy approximate solution of the problem (4.32) under (ii)-differentiability in a similar manner to the aforementioned procedure. So, we have:

$$\begin{cases} \underline{k1}_2^{(1)} = (-\underline{y}_{2n}^{\,r} + \sin(t_n)); \quad \overline{k1}_1^{(1)} = (-\underline{y}_{2n}^{\,r} + \sin(t_n)), \\ \underline{k1}_2^{(2)} = (\underline{y}_{2n}^{\,r} + \cos(t_n) - \sin(t_n)); \quad \overline{k1}_2^{(2)} = (\bar{y}2_n^r + \cos(t_n) - \sin(t_n)). \end{cases} \tag{4.36}$$

Moreover, we obtain

$$\begin{cases} \underline{k3}_2^{(2)} = (1 - h + \frac{h^2}{2})\underline{y}_{2n}^{\,r} - \frac{h^2}{2}\sin(t_n) + (h-1)\sin(t_n + h) + \cos(t_n + h), \\ \overline{k3}_2^{(2)} = (1 - h + \frac{h^2}{2})\bar{y}2_n^r - \frac{h^2}{2}\sin(t_n) + (h-1)\sin(t_n + h) + \cos(t_n + h), \end{cases} \tag{4.37}$$

and

$$\begin{cases} \underline{k4}_2^{(2)} = \underline{y}_{2n}^{\,r} + \frac{h^2}{2}((-10 + 2h - \frac{h^2}{2})\underline{y}_{2n}^{\,r} + (7 - h + \frac{h^2}{2})\sin(t_n) + (2-h)\sin(t_n + \frac{h}{2}) + \sin(t_n + h)), \\ \overline{k4}_2^{(2)} = \bar{y}2_n^r + \frac{h^2}{2}((-10 + 2h - \frac{h^2}{2})\bar{y}2_n^r + (7 - h + \frac{h^2}{2})\sin(t_n) + (2-h)\sin(t_n + \frac{h}{2}) + \sin(t_n + h)). \end{cases} \tag{4.38}$$

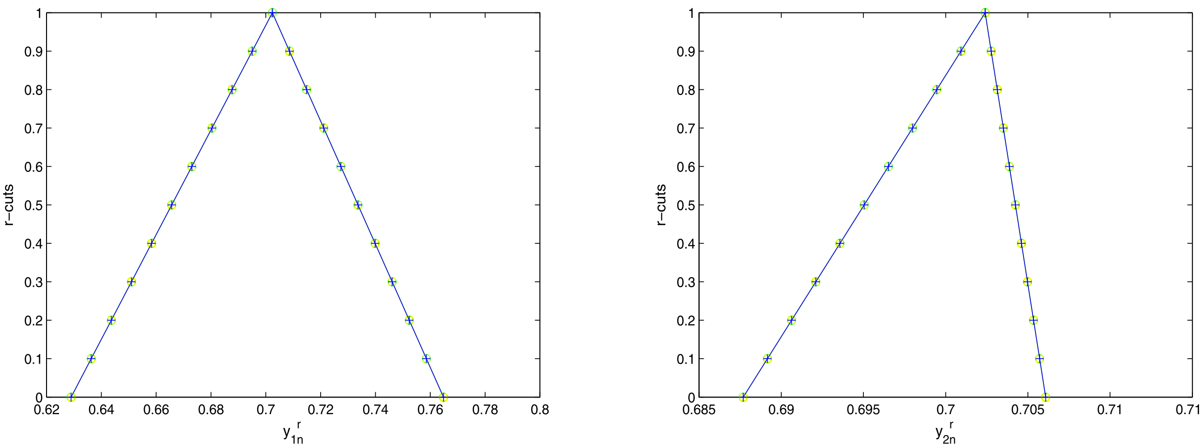
Again, using Eqs. (4.36)–(4.38), the approximate solution $y2_{n+1}^r$ is easily acquired from (3.19).

Table 3
Exact solution (Y_1^r), absolute error function (N_{erk}^r) for the fourth-order ERK method, absolute error function (N_{rk}^r) for the fourth-order RK method under (i)-differentiability at $t = 1$ for $r \in [0, 1]$, Example 4.2.

		ERK4	RK4		ERK4	RK4
r	$\underline{Y_1}^r$	$\underline{N}_{erk}^r$	$\underline{N}_{rk}^r$	$\overline{Y_1}^r$	$\overline{N}_{erk}^r$	$\overline{N}_{rk}^r$
0	0.628928	9.072474e−8	5.031396e−7	0.764842	1.095380e−7	3.989234e−7
0.1	0.636275	9.158906e−8	4.984287e−7	0.758598	1.085210e−7	4.046341e−7
0.2	0.643623	9.245338e−8	4.937177e−7	0.752354	1.075040e−7	4.103448e−7
0.3	0.650970	9.331769e−8	4.890068e−7	0.746110	1.064870e−7	4.160554e−7
0.4	0.658318	9.418201e−8	4.842958e−7	0.739866	1.054700e−7	4.217661e−7
0.5	0.665665	9.504632e−8	4.795849e−7	0.733622	1.044530e−7	4.274768e−7
0.6	0.673013	9.591064e−8	4.748739e−7	0.727379	1.034359e−7	4.331874e−7
0.7	0.680360	9.677496e−8	4.701630e−7	0.721135	1.024189e−7	4.388981e−7
0.8	0.687708	9.763927e−8	4.654520e−7	0.714891	1.014019e−7	4.446088e−7
0.9	0.695055	9.850359e−8	4.607411e−7	0.708647	1.003849e−7	4.503194e−7
1	0.702403	9.936790e−8	4.560301e−7	0.702403	9.936790e−8	4.560301e−7

Table 4
Exact solution (Y_2^r), absolute error function (N_{erk}^r) for the fourth-order ERK method, absolute error function (N_{rk}^r) for the fourth-order RK method under (ii)-differentiability at $t = 1$ for $r \in [0, 1]$, Example 4.2.

		ERK4	RK4		ERK4	RK4
r	$\underline{Y_2}^r$	$\underline{N}_{erk}^r$	$\underline{N}_{rk}^r$	$\overline{Y_2}^r$	$\overline{N}_{erk}^r$	$\overline{N}_{rk}^r$
0	0.687688	1.014039e−7	4.427005e−7	0.706082	9.885889e−8	4.593625e−7
0.1	0.689159	1.012003e−7	4.440334e−7	0.705714	9.8909800e−8	4.590293e−7
0.2	0.690631	1.009967e−7	4.453664e−7	0.705346	9.8960700e−8	4.586960e−7
0.3	0.692102	1.007931e−7	4.466994e−7	0.704978	9.901160e−8	4.583628e−7
0.4	0.693574	1.005895e−7	4.480323e−7	0.704610	9.906250e−8	4.580296e−7
0.5	0.695045	1.003859e−7	4.493653e−7	0.704242	9.911340e−8	4.576963e−7
0.6	0.696517	1.001823e−7	4.506983e−7	0.703875	9.916430e−8	4.573631e−7
0.7	0.697988	9.997871e−8	4.520312e−7	0.703507	9.921520e−8	4.570298e−7
0.8	0.699460	9.977511e−8	4.533642e−7	0.703139	9.926610e−8	4.566966e−7
0.9	0.700931	9.957151e−8	4.546972e−7	0.702771	9.931700e−8	4.563634e−7
1	0.702403	9.936790e−8	4.560301e−7	0.702403	9.936790e−8	4.560301e−7



(a) Fuzzy approximate and exact solutions under (i)- differentiability

(b) Fuzzy approximate and exact solutions under (ii)- differentiability

Fig. 3. A comparison between the fuzzy approximate solution by the fourth-order ERK method (∘), the fourth-order RK method (+) and fuzzy exact solution (.−) at $t = 1$ for $r \in [0, 1]$, Example 4.2.

A comparison between the absolute error functions acquired by our method and the fourth-order RK method described in [1], at $x = 1$ under both types of fuzzy differentiability, is shown in Tables 3 and 4. The results demonstrate that the fourth-order ERK method can solve the problem effectively and achieve higher accuracy in comparison with the fourth-order classical RK method. The approximate solutions obtained by the present method at $x = 1$ are plotted in Fig. 3 under (i) and (ii)-differentiability to make it easier to compare the fuzzy approximate solution of the fourth-order ERK method with the analytic solution and the fourth-order RK method solution. The proposed techniques could achieve the same order of accuracy with a lower number of function evaluations and thus reduces the computational cost in comparison with the fourth-order RK method. From the results, it can be implied that the fuzzy approximate and analytical solutions under (ii)-differentiability are in this case closer to the real solution of the problem compared with the H-differentiability.

Example 4.3. To complete the illustration of our proposed method, the following example is presented. The problem has a switching point on the defined interval. The mixed solutions to the problem are derived according to the type of fuzzy differentiability, which is defined on the neighborhood V of the switching point.

We are concerned with the following FIVP:

$$\begin{cases} y'(t) = (1-t)y(t), \quad 0 \le t \le 2, \\ y(0) = (0, 1, 2), \end{cases} \tag{4.39}$$

where the initial value condition is a triangular symmetric fuzzy number with the parametric form $[r, 2-r]$.

It is obvious that the initial value problem (4.39) on $[0, 1]$ is (i)-differentiable, and at $t = 1$ the problem switches to (ii)-differentiability. So, the point $t = 1$ is a switching point and the obtained solution on $[0, 1]$ is (i)-differentiable and (ii)-differentiable on $[1, 2]$.

Hence, the mixed solution of the problem (4.39) can be obtained by solving

$$\begin{cases} \underline{y}'(t;r) = (1-t)\underline{y}(t;r), \\ \overline{y}'(t;r) = (1-t)\overline{y}(t;r), \\ \underline{y}(0;r) = r, \\ \overline{y}(0;r) = 2-r, \end{cases} \tag{4.40}$$

for $t \in [0, 1]$ under (i)-differentiability and

$$\begin{cases} \underline{y}'(t;r) = (1-t)\underline{y}(t;r), \\ \overline{y}'(t;r) = (1-t)\overline{y}(t;r), \\ \underline{y}(1;r) = re^{\frac{1}{2}}, \\ \overline{y}(1;r) = (2-r)e^{\frac{1}{2}}, \end{cases} \tag{4.41}$$

for $t \in [1, 2]$ under (ii)-differentiability.

It is worth mentioning here that $f'(t, y(t;r)) = [t\underline{y}(t;r)(t-2), t\overline{y}(t;r)(t-2)]$ (see in [10]). Now we use the fourth-order ERK method to obtain the approximate solution of the FIVPs (4.40) and (4.41). Define y_{1_n} to be the approximate value of $x(t_n)$ under (i)-differentiability for $t_n \in [0, 1]$ and y_{2_n} under (ii)-differentiability for $t_n \in [1, 2]$. Then, the fourth-order ERK method is constructed as follows:

$$\begin{cases} \underline{k_1}_1^{(1)} = (1-t_n)\underline{y}_{1_n}; \quad \overline{k_1}_1^{(1)} = (1-t_n)\overline{y}_{1_n}, \\ \underline{k_2}_1^{(1)} = \underline{y}_{1_n}(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n)); \quad \overline{k_2}_1^{(1)} = \overline{y}_{1_n}(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n)), \\ \underline{k_3}_1^{(1)} = \underline{y}_{1_n}(1-(t_n+h))(1+h(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n))) \\ \overline{k_3}_1^{(1)} = \overline{y}_{1_n}(1-(t_n+h))(1+h(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n))) \end{cases} \tag{4.42}$$

and

$$\begin{cases} \underline{k_1}_1^{(2)} = t_n\underline{y}_{1_n}^{r}(t_n-2), \\ \overline{k_1}_1^{(2)} = t_n\overline{y}_{1_n}^{r}(t_n-2). \end{cases} \tag{4.43}$$

In addition, we have:

$$\begin{cases} \underline{k_3}_1^{(2)} = \underline{y}_{1n}^{\,r}(t_n+h)(t_n+h-2)\{1+h(1-t_n-\frac{h}{2})(1+\frac{h}{2}(1-t_n))\}, \\ \overline{k_3}_1^{(2)} = \overline{y}_{1n}^{\,r}(t_n+h)(t_n+h-2)\{1+h(1-t_n-\frac{h}{2})(1+\frac{h}{2}(1-t_n))\}, \end{cases} \tag{4.44}$$

and

$$\begin{cases} \underline{k_4}_1^{(2)} = \underline{y}_{1n}^{\,r}(t_n+\frac{2h}{5})(t_n+\frac{2h}{5}-2)(1+\frac{h}{25}\{7(1-t_n)+(1-(t_n+h))+(2+h)(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n))\}), \\ \overline{k_4}_1^{(2)} = \overline{y}_{1n}^{\,r}(t_n+\frac{2h}{5})(t_n+\frac{2h}{5}-2)(1+\frac{h}{25}\{7(1-t_n)+(1-(t_n+h))+(2+h)(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n))\}), \end{cases} \tag{4.45}$$

where $t_n \in [0, 1]$. To approximate the (ii)-differentiable solution, $y(t_n)$ for $t_n \in [1, 2]$, the fourth-order ERK method is employed by the similar previous process as:

$$\begin{cases} \underline{y_2}_{n+1}(r) = \underline{y_2}_n(r) + h\underline{k_1}_1^{(1)} + \frac{h^2}{8}\underline{k_1}_2^{(2)} + \frac{h^2}{36}\underline{k_3}_2^{(2)} + \frac{25h^2}{72}\underline{k_4}_2^{(2)}, \\ \overline{y_2}_{n+1}(r) = \overline{y_2}_n(r) + h\overline{k_1}_1^{(1)} + \frac{h^2}{8}\overline{k_1}_2^{(2)} + \frac{h^2}{36}\overline{k_3}_2^{(2)} + \frac{25h^2}{72}\overline{k_4}_2^{(2)}, \end{cases} \tag{4.46}$$

where ${k_1}_1^{(1)}, \ldots, {k_4}_2^{(2)}$ are achieved using Eq. (3.20) as:

$$\begin{cases} \underline{k_1}_2^{(1)} = (1-t_n)\underline{y}_{2n}; \quad \overline{k_1}_2^{(1)} = (1-t_n)\overline{y}_{2n}, \\ \underline{k_2}_2^{(1)} = \underline{y}_{2n}(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n)); \quad \overline{k_2}_2^{(1)} = \overline{y}_{2n}(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n)), \\ \underline{k_3}_2^{(1)} = \underline{y}_{2n}(1-(t_n+h))(1+h(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n))) \\ \overline{k_3}_1^{(1)} = \overline{y}_{2n}(1-(t_n+h))(1+h(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n))) \end{cases} \tag{4.47}$$

and

$$\begin{cases} \underline{k_1}_2^{(2)} = t_n\underline{y}_{2n}^{\,r}(t_n-2), \\ \overline{k_1}_2^{(2)} = t_n\overline{y}_{2n}^{\,r}(t_n-2). \end{cases} \tag{4.48}$$

Furthermore, we obtain

$$\begin{cases} \underline{k_3}_2^{(2)} = \underline{y}_{2n}^{\,r}(t_n+h)(t_n+h-2)\{1+h(1-t_n-\frac{h}{2})(1+\frac{h}{2}(1-t_n))\}, \\ \overline{k_3}_2^{(2)} = \overline{y}_{2n}^{\,r}(t_n+h)(t_n+h-2)\{1+h(1-t_n-\frac{h}{2})(1+\frac{h}{2}(1-t_n))\}, \end{cases} \tag{4.49}$$

and

$$\begin{cases} \underline{k_4}_2^{(2)} = \underline{y}_{2n}^{\,r}(t_n+\frac{2h}{5})(t_n+\frac{2h}{5}-2)(1+\frac{h}{25}\{7(1-t_n)+(1-(t_n+h))+(2+h)(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n))\}), \\ \overline{k_4}_2^{(2)} = \overline{y}_{2n}^{\,r}(t_n+\frac{2h}{5})(t_n+\frac{2h}{5}-2)(1+\frac{h}{25}\{7(1-t_n)+(1-(t_n+h))+(2+h)(1-(t_n+\frac{h}{2}))(1+\frac{h}{2}(1-t_n))\}). \end{cases} \tag{4.50}$$

The exact solution and approximate solutions obtained by using the fourth-order ERK method and the fourth-order RK method are plotted in Fig. 4. In addition, the results are shown in Tables 5 and 6 under (i) and (ii)-differentiability, respectively. In Fig. 5, we display the r-level sets of the exact solution of the problem (4.39) for $r \in [0, 1]$.

5. Conclusion

In this paper, a two-step RK formula of order four with a reduced number of function evaluations is discussed. A clear advantage of this technique lies in the fact that an explicit RK method of order four requires four function evaluations, whereas our proposed RK method needs only three function evaluations of f and f' per step for autonomous systems. This leads to an enhancement of the order of the accuracy of the solutions and a reduction in the computational cost, especially for FODEs that needs to solve two ODEs systems simultaneously in order to achieve the fuzzy approximate solution.

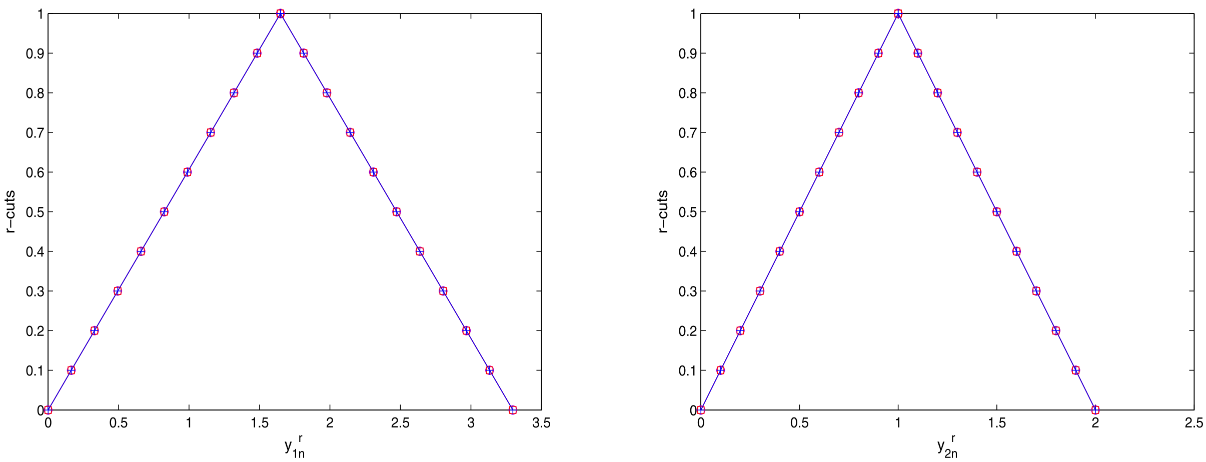
On the other hand, we have developed an adaptive version of the fourth-order ERK method for non-autonomous FODEs systems. We have presented a number of examples that solve this type of FODEs system. Practically, we have derived suitable error estimate of the local error, and provided fuzzy differentiability-choosing strategies for this

Table 5
Exact solution (Y_1^r), absolute error function (N_{erk}^r) for the fourth-order ERK method, absolute error function (N_{rk}^r) for the fourth-order RK method under (i)-differentiability at $t = 1$ for $r \in [0, 1]$, Example 4.3.

		ERK4	RK4		ERK4	RK4
r	$\underline{Y_1}^r$	$\underline{N}_{erk}^r$	$\underline{N}_{rk}^r$	$\overline{Y_1}^r$	$\overline{N}_{erk}^r$	$\overline{N}_{rk}^r$
0	0	0	0	3.297442	7.867828e−7	5.272934e−7
0.1	0.164872	3.933914e−8	2.636467e−8	3.132570	7.474436e−7	5.009287e−7
0.2	0.329744	7.867828e−8	5.272934e−8	2.967698	7.081045e−7	4.745641e−7
0.3	0.494616	1.180174e−7	7.909401e−8	2.802826	6.687653e−7	4.481994e−7
0.4	0.659488	1.573565e−7	1.054586e−7	2.637954	6.294262e−7	4.218347e−7
0.5	0.824360	1.966957e−7	1.318233e−7	2.473081	5.900871e−7	3.954700e−7
0.6	0.989232	2.360348e−7	1.581880e−7	2.308209	5.507479e−7	3.691054e−7
0.7	1.154104	2.753739e−7	1.845527e−7	2.143337	5.114088e−7	3.427407e−7
0.8	1.318977	3.147131e−7	2.109173e−7	1.978465	4.720696e−7	3.163760e−7
0.9	1.483849	3.540522e−7	2.372820e−7	1.813593	4.327305e−7	2.900114e−7
1	1.648721	3.933914e−7	2.636467e−7	1.648721	3.933914e−7	2.636467e−7

Table 6
Exact solution (Y_2^r), absolute error function (N_{erk}^r) for the fourth-order ERK method, absolute error function (N_{rk}^r) for the fourth-order RK method under (ii)-differentiability at $t = 1$ for $r \in [0, 1]$, Example 4.3.

		ERK4	RK4		ERK4	RK4
r	$\underline{Y_2}^r$	$\underline{N}_e^r$	$\underline{N}_e^r$	$\overline{Y_2}^r$	$\overline{N}_e^r$	$\overline{N}_e^r$
0	0	0	0	2.000000	5.557013e−7	2.198941e−7
0.1	0.100000	2.778506e−8	1.099470e−8	1.900000	5.279162e−7	2.088994e−7
0.2	0.200000	5.557013e−8	2.198941e−8	1.800000	5.001311e−7	1.979047e−7
0.3	0.300000	8.335519e−8	3.298412e−8	1.700000	4.723461e−7	1.869100e−7
0.4	0.400000	1.111402e−7	4.397883e−8	1.600000	4.445610e−7	1.759153e−7
0.5	0.500000	1.389253e−7	5.497353e−8	1.500000	4.167759e−7	1.649206e−7
0.6	0.600000	1.667103e−7	6.596824e−8	1.400000	3.889909e−7	1.539259e−7
0.7	0.700000	1.944954e−7	7.696295e−8	1.300000	3.612058e−7	1.429312e−7
0.8	0.800000	2.222805e−7	8.795766e−8	1.200000	3.334207e−7	1.319364e−7
0.9	0.900000	2.500655e−7	9.895236e−8	1.100000	3.056357e−7	1.209417e−7
1	1.000000	2.778506e−7	1.099470e−7	1.000000	2.778506e−7	1.099470e−7



(a) Fuzzy approximate and exact solutions under (i)- differentiability

(b) Fuzzy approximate and exact solutions under (ii)- differentiability

Fig. 4. A comparison between the fuzzy approximate solution by the fourth-order ERK method (∘), the fourth-order RK method (+) and fuzzy exact solution (□) at $t = 1$ for $r \in [0, 1]$, Example 4.3.

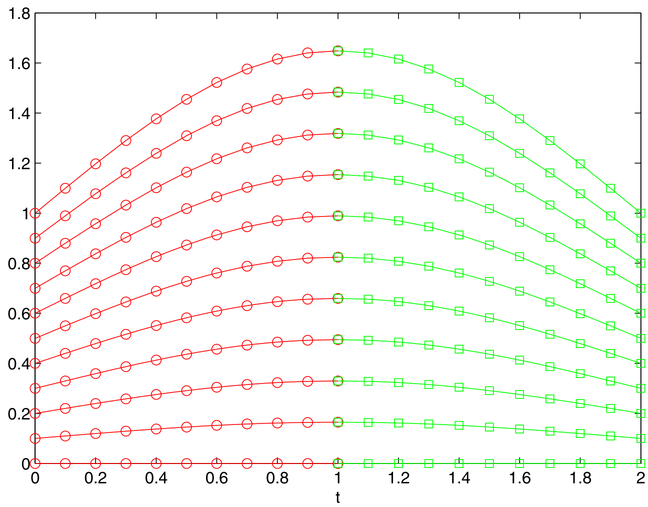


Fig. 5. Exact solution to the problem (4.39) that is (i)-differentiable on $t \in [0, 1)$, (∘-red line), and (ii)-differentiable on $t \in (1, 2]$, (□-green line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

type of FODEs system. In fact, the procedure consists of using a fourth-order ERK method with local truncation error of order four with respect to the type of fuzzy differentiability. Furthermore, according to Bede et al. [17] "The importance of converting a FDE to a system of ODEs is that then any suitable numerical method for ODEs may be implemented". In this research, we have solved FDEs under generalized H-differentiability by applying the fourth order ERK method, and provided the convergence theorem as well. We have also studied the numerical solutions of the FODE system with a switching point, where the piecewise solution changes from (i)- to (ii)-differentiability. We have presented an example to illustrate how this type of FDE can be solved by using the proposed method.

To summarize, we briefly state the following objectives which have been achieved:

(I) The given non-autonomous or autonomous FODE has been solved using the fourth-order ERK method, with lower computational cost, especially for cases where f' is not more expensive to evaluate than f, in comparison with the fourth-order classical RK method.

(II) The method has been derived under both types of fuzzy derivatives, and all the examples have been tested under (i) and (ii)-differentiability. However, the outcomes illustrate that the case of (i)-differentiability does not have adequate efficiency for the numerical solutions of FODEs.

Future research will attempt to apply the RK Nystrom method for solving second order FODEs under generalized H-differentiability. In addition, the numerical methods for solving fuzzy fractional differential equations will be studied.

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